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OBSERVATIONS OF THE SEVEN INNER SATELLITES OF *SATURN*,

By E. E. BARNARD.

The following measures of the satellites of *Saturn* are continued from *Astronomical Journal*, Vol. XXVII, No. 639-40. In the measures of *Hyperion* there are several cases where uncertainty exists as to whether the object observed was not a fixed star. The observations are in Central Standard Time, $6^h\ 0^m$ slow of Greenwich Mean Time. All the measures are double distances. The observations are not corrected for refraction.

The Measures of the Satellites.

1912-13	C.S. Time	P.A.	Dist.	Cps.	P.A. of Wires	Remarks
	h m s	°	"		°	
			Tethys and *Mimas*.			
Mar. 4	6 46 15	220.00	.....	5	....	Seeing good. *Mimas* close to the ring.
	6 51 10	.....	17.68	8	130.1	
			Dione and *Mimas*.			
Nov. 17	10 59 4	252.04	.....	5	....	
	11 3 20	.....	27.88	8	161.6	
Feb. 8	9 18 21	95.95	.....	5	....	Seeing good.
	9 22 51	.....	73.08	8	5.8	
			Rhea and *Mimas*.			
Jan. 28	8 6 23	113.19	.....	5	....	Fairly well seen but close to ring.
	8 10 56	.....	22.73	9	23.7	
			Tethys and *Enceladus*.			
Oct. 1	15 38 24	280.11	.....	5	....	
	15 42 58	.....	78.22	8	190.4	
Oct. 15	16 19 56	159.07	.....	5	....	Seeing fair.
	16 24 32	.....	36.26	8	69.2	
Oct. 29	11 44 55	86.95	.....	5	....	
	11 48 11	.....	35.28	2	176.5	Clouds. Single distances
	13 7 9	80.69	.....	5	....	Very faint from very bad seeing.
	13 11 54	.....	37.00	8	170.5	Wind very high.
Nov. 16	9 58 15	183.05	.....	5	....	
	10 1 58	.....	30.00	8	91.7	
Nov. 17	13 15 8	287.69	.....	5	....	
	13 18 24	.....	16.84	8	17.6	
Dec. 28	11 13 59	147.62	.....	5	....	Seeing excessively bad.
	11 18 8	.....	16.96	9	58.3	
Dec. 31	10 33 54	250.34	.....	5	....	Seeing good.
	10 38 18	.....	56.00	8	160.5	

Tethys AND *Enceladus.* (Continued.)

1912-13	C.S. Time	P.A.	Dist.	Cps.	P.A. of Wires	Remarks
	h m s	°	″		°	
Jan. 18	10 34 52	63.64	. . .	5	. . .	
	10 39 9	. . .	32.60	8	153.6	
Mar. 4	6 56 39	210.44	.	5	. . .	
	7 1 32		30.47	8	120.1	

Dione AND *Enceladus.*

1912-13	C.S. Time	P.A.	Dist.	Cps.	P.A. of Wires	Remarks
Sept. 8	13 29 12	94.58	. . . . .	5	. .	
	13 33 28	. . .	27.86	8	185.2	
Sept. 21	13 56 5	105.04	. . . .	5	.	
	13 59 36	. . . .	49.16	8	15.0	
Sept. 22	13 46 42	79.69	.	5		
	13 50 37	. . .	52.57	8	169.8	
Oct. 19	12 56 14	86.04	. .	5	. .	
	13 0 43	. .	23.60	8	175.8	
Nov. 24	10 16 0	81.97	. . . . .	5	.	Very difficult from bad seeing.
	10 22 14	. . .	34.44	8	171.8	
Dec. 21	9 44 48	71.79	. . . .	5	. . .	
	9 50 4	. .	31.44	10	161.7	
Jan. 28	7 16 1	148.78	. . . .	5	. . . .	
	7 19 34	. . . .	30.51	8	58.7	
Feb. 4	6 1 20	324.05	. . . .	5	. .	Seeing fair.
	6 5 30	. . .	33.61	8	54.0	
Feb. 8	9 54 57	87.84	. . . . .	5	. .	Seeing good.
	9 59 8	. . .	65.91	8	177.8	

Rhea AND *Enceladus.*

1912-13	C.S. Time	P.A.	Dist.	Cps.	P.A. of Wires	Remarks
Aug. 20	15 17 24	78.67	. . . . .	5	. . . .	Faint in clouds. Seen only once
	15 23 46	. . . .	65.34	8	168.6	in a while.
Oct. 12	15 10 0	143.15	. . . .	5	. . .	Seeing bad.
	15 15 20	. . . . .	30.20	8	52.7	
Dec. 8	11 42 6	276.91	. . . . .	5	. . .	
	11 46 40	. . . . .	46.86	8	7.1	
Jan. 12	11 6 30	333.32	. . .	5	. . . .	Excessively faint in clouds. Dis-
	11 13 40	. . . . .	35.89	7	63.0	tances discordant by 2″.
Feb. 1	9 30 50	233.70	. . . . .	5	. . .	
	9 35 54	. .	31.03	8	143.7	
Feb. 9	9 38 5	287.73	. . . .	5	. . . .	
	9 42 36	.	35.66	8	17.8	

Dione AND *Tethys.*

1912-13	C.S. Time	P.A.	Dist.	Cps.	P.A. of Wires	Remarks
Sept. 1	13 45 59	259.48	. . . . .	5	. . . .	Seeing excessively bad. Clouds.
	13 50 44	. . . . .	13.46	10	169.4	
Sept. 8	13 37 8	104.82	. . . . .	5	. . . .	
	13 41 32	. . . . .	37.87	8	14.7	

Dione AND *Tethys*. (Continued.)

1912-13	C.S. Time	P.A.	Dist.	Cps.	P.A. of Wires	Remarks
	h m s	°	″		°	
Sept. 21	13 48 8	218.64	.....	5	....	
	13 52 10	.....	42.56	8	129.0	
Sept. 22	13 37 20	64.55	.....	5	....	
	13 41 30	.....	80.93	8	154.3	
Oct. 1	15 29 56	247.47	.....	5	....	
	15 34 2	.....	22.94	8	157.4	
Oct. 19	13 5 39	55.11	.....	5	....	Seeing poor towards last. Good at first.
	13 10 6	.....	31.12	8	145.3	
Nov. 15	11 32 54	157.24	.....	6	....	
	11 37 37	.....	40.60	8	67.0	
Nov. 17	13 7 45	227.96	.....	5	....	
	13 11 24	.....	61.10	8	137.6	
Nov. 24	9 52 36	9.05	.....	5	....	Seeing very bad.
	9 57 16	.....	33.75	8	99.2	
Dec. 21	9 35 17	112.05	.....	5	....	
	9 39 50	.....	50.12	8	22.2	
Dec. 22	10 53 51	308.77	.....	5	....	
	10 58 20	.....	60.40	8	38.6	
Dec. 28	11 4 19	266.24	.....	5	....	Seeing excessively bad.
	11 8 20	.....	101.28	8	176.3	
Dec. 31	10 57 46	136.15	.....	5	....	
	11 1 43	.....	27.24	8	46.0	
Jan. 4	10 46 55	64.07	.....	5	....	
	10 51 34	.....	82.24	8	154.1	
Jan. 18	10 43 50	112.13	.....	5	....	
	10 48 8	.....	46.55	8	22.1	
Jan. 25	10 51 20	97.48	.....	5	.	Stopped by clouds. One rough setting.
	10 53 52	...	$16\frac{1}{2}\pm$	1		
Jan. 28	7 23 13	191.12	.....	5	.	Seeing fair.
	7 26 59	.....	53.59	8	100.7	
Feb. 8	10 3 0	138.73	.....	5	....	Seeing good.
	10 6 44	.....	37.84	8	48.8	
Mar. 4	7 8 32	269.07	.....	5	....	
	7 12 48	.....	54.42	8	179.1	
Mar. 18	6 43 44	201.25	.....	5	....	Very difficult. Excessively bad seeing.
	6 48 20	.....	44.11	8	179.1	

Rhea AND *Tethys*.

1912-13	C.S. Time	P.A.	Dist.	Cps.	P.A. of Wires	Remarks
Aug. 13	15 58 49	225.64	.....	5	....	Seeing fair.
	16 2 34	.....	34.13	8	135.9	
Aug. 20	15 8 0	68.29	.....	5	....	
	15 12 9	.....	29.29	8	158.6	

Rhea and *Tethys*. (Continued.)

1912-13	C.S. Time (h m s)	P.A.	Dist. (″)	Cps.	P.A. of Wires (°)	Remarks
Aug. 24	13 52 40	101.74	. . .	5	.	Seeing poor. Planet low.
	13 57 8	.	77.78	9	11.8	
Sept. 1	13 36 29	123.58		5		Seeing bad.
	13 41 34	.	66.38	8	33.1	
Sept. 7	13 24 39	68.23	.	5	. . .	Seeing poor.
	13 29 34	. .	103.94	8	158.0	
Sept. 21	13 39 22	326.53	. .	5	.	
	13 43 52	. .	30.53	8	56.5	
Oct. 12	15 20 33	132.89	. .	6	.	Seeing bad.
	15 25 32		68.53	8	42.7	
Oct. 15	16 10 31	272.79		5	. . .	Seeing fair.
	16 14 50	. .	110.67	8	2.7	
Oct. 29	11 25 1	247.69	.	5		
	11 39 44	. . .	109.85	8	157.5	
Nov. 9	13 22 0	73.76	.	5		Seeing very bad. Through dense haze.
	13 26 20	.	36.80	8	163.7	
Nov. 10	12 6 21	47.38	. .	5	. . . .	Seeing excessively bad.
	12 10 56	. . . .	60.02	8	137.6	
Nov. 15	11 15 49	275.22	.	5		
	11 20 46	.	86.31	8	5.0	
Nov. 16	9 49 2	270.01		5	.	
	9 53 34	.	53.04	8	179.7	
Nov. 24	9 42 29	295.99	.	5	.	
	9 47 48	.	96.73	8	25.8	
Dec. 8	11 51 39	259.84	. . .	5	.	
	11 55 22	. .	56.77	8	169.1	
Dec. 21	9 25 37	282.58	. . .	5	. . .	Excessively bad seeing.
	9 30 11	.	56.48	8	12.7	
Dec. 31	10 49 1	255.41	.	5		
	10 52 58		36.11	8	165.5	
Jan. 4	10 36 6	301.35	.	6	. . .	
	10 41 11	. . . .	55.21	8	31.1	
Jan. 12	11 27 20	340.06	. .	5	. .	
	11 32 40	. . .	48.84	9	69.8	
Jan. 18	10 52 20	247.32	. .	5	. . . .	
	10 56 56	. . . .	90.64	8	118.2	
Feb. 1	9 40 52	208.11	. . . . .	5	.	
	9 45 48	. . . . .	34.01	8	157.1	
Feb. 4	6 10 49	309.02	. .	5	. .	Seeing fair.
	6 15 36	. .	59.45	8	39.0	

Rhea AND *Tethys*. (Continued.)

1912-13	C.S. Time	P.A.	Dist.	Cps.	P.A. of Wires	Remarks
	h m s	°	″		°	
Feb. 9	9 19 59	311.17	. . . .	5	. .	Seeing excessively bad.
	9 24 32	. . . . .	56.16	8	41.3	
Feb. 11	6 9 56	104.82	.	5	.	Seeing bad.
	6 24 36	. . .	50.82	8	14.5	
Mar. 18	6 53 6	249.51	. .	5	. . .	Very difficult. Excessively bad seeing.
	6 57 40	. . . .	87.66	8	159.7	
			Titan AND *Tethys*.			
Oct. 15	16 29 40	128.69	. . . . .	5	. . . .	Seeing fair.
	16 34 10	. . . . .	82.34	8	38.7	
Jan. 18	11 1 16	157.54	. .	5	. .	Seeing good.
	11 5 39	. . . .	107.39	8	67.6	
Feb. 4	6 29 43	135.33	.	5	. . . .	Seeing fair.
	6 34 17	. . . . .	78.71	8	45.5	
			Rhea AND *Dione*.			
Aug. 13	15 50 33	248.87	. . . . .	5	. . .	Seeing fair.
	15 54 38	. . .	18.47	8	158.9	
Aug. 20	14 58 31	59.21	. . . .	5	. . . .	
	15 3 32	. . .	15.64	8	148.6	
Sept. 7	13 34 11	63.66	. . . . .	5	. . . .	
	13 38 36	. . . .	104.01	8	153.5	
Sept. 8	13 45 22	282.49	. . . . .	5	. . .	Seeing bad.
	13 49 48	. . .	95.16	8	12.7	
Sept. 22	13 27 22	265.83	. . .	5	. .	
	13 32 22	. . . . .	108.32	8	175.8	
Oct. 1	15 21 58	302.07		5	. . . .	
	15 26 2	. . . . .	19.98	8	32.4	
Oct. 12	15 31 2	100.86	.	5	. . . .	Seeing bad.
	15 36 22	. . . . .	122.68	8	10.7	
Oct. 15	16 1 3	279.87	. . . . .	5	. . . .	Seeing fair.
	16 5 52	. . .	35.39	8	10.2	
Oct. 19	12 42 43	277.09	. . .	5	. . .	
	12 50 10	. . . .	132.40	8	7.1	
Oct. 29	11 26 25	269.45	. .	5	. . .	
	11 30 33	. . . . .	35.73	8	179.5	
Nov. 9	13 30 29	68.33	. .	5	. . .	Seeing very poor.
	13 35 25	. . . . .	111.92	8	158.3	
Nov. 10	11 57 51	292.25	. . . . .	5	. . . .	
	12 2 6	. .	72.24	8	22.1	
Nov. 16	9 38 17	246.51	. . . . .	5	. . . .	
	9 43 42	. . . . .	110.52	8	156.7	

Rhea AND *Dione*. (Continued.)

1912-13	C.S. Time	P.A.	Dist.	Cps.	P.A. of Wires	Remarks
	h m s	°	″			
Nov. 17	12 59 27	98.57	....	5	....	
	13 3 41	.....	86.10	8	8.6	
Nov. 24	9 29 12	275.65	....	5	....	Seeing excessively bad.
	9 35 7	....	93.86	8	5.8	
Nov. 28	11 14 38	44.13	....	6	....	Clouds.
Dec. 8	11 59 12	255.14	....	5	....	
	12 3 26	....	77.89	8	165.1	
Dec. 22	10 45 14	230.89	....	5	....	
	10 49 42	....	48.82	8	141.2	
Jan. 12	11 38 34	300.79	....	5	....	Excessively difficult. Dense haze and excessively bad seeing.
	11 43 35	....	42.40	8	30.8	
Jan. 25	10 20 57	53.49	....	5	....	
	10 46 16	....	66.72	10	142.1	
Jan. 28	7 7 38	74.93	.....	5	....	Seeing very poor.
	7 12 18	....	54.93	8	164.7	
Feb. 1	9 49 56	210.17	....	5	....	
	9 54 30	....	50.31	8	120.2	
Feb. 4	6 20 45	65.77	....	5	....	Seeing fair.
	6 25 26	.....	46.49	8	155.5	
Feb. 8	10 10 57	344.02	.....	5	....	Seeing good.
	10 14 40	....	43.90	8	73.3	
Feb. 9	9 28 41	265.62	....	5	....	Seeing very bad.
	9 33 22	....	58.17	8	175.8	
Feb. 11	6 11 19	229.49	....	5	....	Seeing bad.
	6 16 10	....	37.12	8	139.5	
Mar. 4	7 17 1	303.03	....	5	....	
	7 21 31	....	44.83	8	33.1	

Titan AND *Dione*.

1912-13	C.S. Time	P.A.	Dist.	Cps.	P.A. of Wires	Remarks
Sept. 22	13 55 32	298.22	....	5	....	Seeing fair to good.
	13 59 37	....	80.01	8	27.8	
Oct. 12	15 42 17	229.52	....	5	....	Seeing bad.
	15 47 22	....	100.01	8	139.7	
Oct. 19	13 15 42	69.31	....	5	....	Seeing poor.
	13 20 26	....	121.02	8	159.3	
Nov. 15	11 42 10	204.42	....	5	....	
	11 47 6	....	71.68	8	114.0	
Dec. 21	9 55 37	80.69	....	5	....	
	10 1 6		134.38	8	170.7	
Dec. 22	11 3 18	79.13	....	5		
	11 9 0	....	212.80	8	169.2	

Titan AND *Dione*. (Continued.)

1912-13	C.S. Time h m s	P.A. °	Dist. ″	Cps.	P.A. of Wires °	Remarks
Dec. 28	11 41 10	279.31	.....	5	...	
	11 45 10	....	152.49	8	9.3	
Jan. 28	7 31 43	295.74	.....	5	....	
	7 36 25	.....	165.67	8	25.7	
Feb. 8	10 20 0	54.38	....	5	...	
	10 24 22	.....	112.28	8	144.3	

Titan AND *Rhea*.

1912-13	C.S. Time h m s	P.A. °	Dist. ″	Cps.	P.A. of Wires °	Remarks
Aug. 20	15 33 11	287.47	.....	5	...	
	15 39 19	.....	136.83	9	17.6	
Oct. 29	11 17 43	183.89	..	5	....	
	11 22 14	....	57.97	8	93.5	
Nov. 10	12 21 23	264.85	..	5	....	
	12 27 12	....	196.34	8	174.6	
Nov. 16	10 6 4	107.13	......	5	...	
	10 10 59	.....	177.78	10	17.2	
Nov. 24	10 1 42	307.02	.....	5	....	Seeing very bad.
	10 6 38	....	70.54	9	36.8	
Dec. 8	12 7 16	56.46	.....	5	..	
	12 12 15	....	145.70	8	146.1	
Dec. 31	11 5 42	211.70	....	5	..	Seeing bad.
	11 10 18	.....	66.45	8	122.0	
Feb. 1	9 58 59	228.54	.	5	.	Seeing fair.
	10 3 12	..	64.00	8	138.2	
Feb. 9	9 48 27	70.63	.....	5	....	Seeing excessively bad.
	9 53 24	.....	158.54	8	160.8	
Feb. 11	6 29 15	324.44	.....	5	....	Seeing bad.
	6 34 16	....	128.14	8	54.5	
Mar. 4	7 25 53	236.64	.....	5	....	
	7 30 36	.....	93.83	8	146.6	

Rhea AND *Hyperion*.

1912-13	C.S. Time h m s	P.A. °	Dist. ″	Cps.	P.A. of Wires °	Remarks
Oct. 1	15 52 21	92.94	.....	5	..	Seeing very bad.
	15 58 22	.....	143.00	8	2.9	
Oct. 12	15 57 47	255.34	.....	5	....	Seeing bad.
	16 4 40	.....	205.44	8	165.2	
Nov. 16	10 21 20	53.76	.....	5	....	A fainter star 1′ south.
	10 25 45	.....	90.98	8	143.7	
Feb. 11	7 12 18	45.49	.....	6	....	14 mag. ? if *Hyperion*. Difficult from brightness of *Saturn*. Seeing poor. Some question of a very large error in the angular setting of wires for distances.
	7 19 36	.....	77.14	8	23.5	

Rhea AND *Hyperion.* (Continued.)

1912-13	C.S Time h m s	P.A.	Dist. "	Cps.	P.A. of Wires	Remarks
Mar. 4	8 1 33	358.44	. . .	5	. .	13.8 mag. ? if *Hyperion.*
	8 5 34		73.15	8	88.6	

Titan AND *Hyperion.*

1912-13	C.S Time	P.A.	Dist.	Cps.	P.A. of Wires	Remarks
Aug. 20	15 58 22	60.65	. . .	5	.	
	16 3 38		149.58	8	150.6	
Aug. 24	13 32 49	86.62	. . .	5	. . . .	Seeing poor. Planet low. *Hyperion* v. faint and difficult.
	13 41 59	.	91.04	9	176.8	
Oct. 29	12 48 46	270.87	. . . .	5	.	
	12 56 18	.	174.80	9	1.0	
Nov. 9	13 48 35	224.20	. .	5	.	? if *Hyperion.* A fainter object 1′ north following. ? if this last was not *Hyperion.* Lost in haze.
	13 54 8	. . .	170.94	8	134.2	
Nov. 15	13 11 15	97.77	. . . .	5	. .	? if *Hyperion.*
	13 17 29	. . .	221.60	8	7.8	

Rhea AND *a star* (?).

1912-13	C.S Time	P.A.	Dist.	Cps.	P.A. of Wires	Remarks
Dec. 28	11 22 48	173.03	. . . .	5	. .	Seeing very bad. 13 mag.
	11 26 55	. . . . .	29.02	8	82.3	
Feb. 11	6 48 34	16.59	.	5	.	14.5 mag. Very difficult. ? if *Hyperion.* A brighter star 1½′ north and fol. another 2′ fol.
	6 56 11	. . .	164.02	10	106.5	
Mar. 4	7 46 23	341.05	. .	5		14 mag. ? if *Hyperion.*
	7 51 3	. . . .	90.17	8	71.1	

Tethys AND *a star* (?).

1912-13	C.S Time	P.A.	Dist.	Cps.	P.A. of Wires	Remarks
Dec. 28	11 31 25	229.75	. . . .	5	.	Seeing very bad.
	11 35 50		19.88	8	139.8	

Dione AND *a star* (?).

1912-13	C.S Time	P.A.	Dist.	Cps.	P.A. of Wires	Remarks
Feb. 4	6 53 38	119.81	. . . .	5		13.7 mag. Very difficult. Seeing very bad.
	7 0 2	. . . .	114.89	11	29.5	

ESTIMATED MAGNITUDE OF THE SATELLITES.

1912-13	*Mimas*	*Enceladus*	*Tethys*	*Dione*	*Rhea*	*Titan*	*Hyperion*	Remarks
Aug. 13		.	12.6	12½	12	.	. . .	
20		.	12.4	12.3	11.8	.	14	*Titan* is yellow.
24		.	13	12.5	12	.	.	
Sept. 1			12.2	12.3	12			
7			12.8	12.3	11½		.	
8		14	13	12.2	12			
21		13½	12	13	12			
22		13	12.2	12.2	12		. .	
Oct. 1		13	12.5	11.7	11.0		. .	
12		13½	13	12.1	12	. .	13.9	
15		14	12.7	12.7	11½			
19		14	12	12.1	11			
29		13	12.2	12½	12		15	

1912-13	*Mimas*	*Enceladus*	*Tethys*	*Dione*	*Rhea*	*Titan*	*Hyperion*	Remarks
Nov. 15	....	....	12½	12½	12	..	13	? if *Hyperion*.
16	....	13½	13	13	12	..	14	
17	14½	13.5	12.5	12.1	11.9	..	...	*Titan* a rich orange yellow
24	...	.	12.4	12.6	12	..	...	
Dec. 8	..	13	12½	12½	12	..	.	
21	..	..	13	13	12	..	...	*Enceladus* faint.
22		...	12.5	12	10-11	.	...	
28		14?	12	12	11.5	.	...	
31		14	12.6	12.6	12	.	...	
Jan. 12	.	13	12½	12½	12	..	...	
18		13	12½	12½	12	.	...	
25		....	13	13	12.5	.	...	
28		13	12½	12	12½	.	..	
Feb. 1		13	12	12½	12		..	
4		13	12	12.5	12		.	
8	15	13.2	12.3	12.3	12		.	
9		14	13	13	12	.	..	
11	.	..	12.5	12.7	12	9.5	.	*Titan* very yellow.
Mar. 4	14	13½	12.3	12.3	12	.		
Mar. 18		...	12½	12½	12		.	

The estimated magnitudes are taken from a sketch, showing the positions of the satellites, made just before beginning the measures on each date. The nearer satellites are subject to much change in apparent brightness when close to the planet or rings.

Yerkes Observatory, Williams Bay, Wisconsin, 1913 March 26.

OBSERVATIONS OF COMETS,

MADE WITH THE 26-INCH AND 12-INCH EQUATORIALS OF THE U. S. NAVAL OBSERVATORY,

[Communicated by Capt. J. L. Jayne, U. S. Navy, Superintendent].

Date Wash. M.T.	⁎	Comp.	$\Delta\alpha$	$\Delta\delta$	App. α	App. δ	log $p\Delta$ α	log $p\Delta$ δ	Red. to App. Place α	Red. to App. Place δ
			Comet 1912 *a* (Gale)							
1912 h m s			m s	′ ″	h m s	° ′ ″			s	″
Sept. 28 7 8 47	1	25 5	−2 18.96	− 2 52.6	15 10 58.10	− 9 30 48.4	9.630	0.768	+1.12	− 9.9*
30 7 18 39	2	24 , 8	−0 54.83	+10 13.4	15 16 51.11	− 6 49 25.5	9.636	0.759	+1.11	− 9.2*
Oct. 4 7 11 5	3	25 , 5	+3 0.91	+ 0 27.6	15 26 57.29	− 1 42 22.1	9.631	0.746	+1.11	− 8.2*
12 6 55 5	4	5 , 1	+3 22.07	+16 47.6	15 41 56.52	+ 7 26 30.8	9.635	0.711	+1.01	− 6.7*
15 7 26 57	5	30 , 6	+1 6.32	+ 3 20.5	15 46 9.74	+10 28 43.3	9.660	0.714	+0.98	− 6.2*
26 6 18 59	6	15 , 5	−0 51.23	+ 3 54.2	15 57 53.38	+20 9 6.6	9.661	0.640	+0.78	− 6.1*
1913 Jan. 9 6 37 47	7	25 , 5	+2 21.34	+ 2 16.2	18 12 25.74	+75 40 6.8	0.223	0.622	−5.37	−14.4†
14 7 36 44	8	25 , 5	+5 19.98	− 5 55.9	18 55 34.55	+79 31 39.7	0.318	0.680	−8.12	−12.5†
Mar. 5 10 25 19	9	25 , 5	+2 34.46	− 5 51.3	4 55 6.44	+65 8 24.7	9.996	9.374*n*	+0.56	+24.0†
28 9 38 35	10	25 , 5	−2 21.06	+ 2 33.3	5 37 8.77	+56 56 14.6	9.885	9.837	+0.49	+20.8†
Apr. 5 9 36 4	11	30 , 6	+1 0.01	− 6 22.3	5 49 36.91	+54 43 32.3	9.873	0.113	+0.36	+19.9†
			Comet 1912 *b*							
1912 Nov. 9 17 30 52	12	25 , 5	−2 11.53	− 0 3.	11 3 37.05	−25 7 49.9	9.450*n*	0.869	+1.33	− 0.4†
			Comet 1912 *c* (Borrelly)							
1912 Nov. 5 6 28 19	13	30 , 6	+1 12.68	− 4 7.1	18 12 17.26	+33 41 32.5	9.617	0.316	+0.56	+ 4.6*
8 6 55 32	14	25 , 5	−1 33.95	− 8 26.3	18 32 15.29	+28 50 31.0	9.623	0.456	+0.77	+ 5.6*
16 8 6 55	15	12 , 4	+0 42.06	− 2 3.9	19 11 58.26	+17 31 44.6	9.652	0.655	+1.20	+ 5.7*

Observers: *C. B. Watts, †H. E. Burton.

The first six observations of 1912 *a* and those of 1912 *c* were made with the 12-inch equatorial; the last five of 1912 *a* and the one of 1912 *b* with the 26-inch equatorial.

Mean Places of the Comparison-Stars for the beginning of the year.

*	α	δ	Authority	*	α	δ	Authority
	h m s	° ′ ″			h m s	° ′ ″	
1	15 13 45.94	− 9 27 45.9	A. G. Wien-Ottakring 5345	9	4 52 31.42	+65 13 52.0	A.G. Christiania 809
2	15 17 44.83	− 6 59 29.7	A. G. Wien-Ottakring 5368	10	5 39 29.35	+56 53 20.4	Boss Preliminary Gen. Cat. 1412
3	15 23 55.27	− 1 42 41.5	A.G. Nicolajew 3937	11	5 48 36.54	+54 49 34.7	A. G. Cambridge U.S., 2337
4	15 38 33.44	+ 7 9 49.9	A.G. Leipzig II 7056	12	11 5 47.25	−25 7 46.5	Argentine General Cat. 15286
5	15 45 2.44	+10 25 29.0	A.G. Leipzig I 5507	13	18 11 4.02	+33 45 35.0	A.G. Leiden 6563
6	15 58 13.83	+20 5 18.5	A.G. Berlin B. 5492	14	18 33 48.47	+28 58 51.7	A. G. Cambridge (Engl.) 9077
7	18 10 9.77	+75 38 5.0	A.G. Kasan 3047	15	19 11 15.00	+17 33 42.8	A.G. Berlin A. 7329
8	18 50 22.69	+79 37 48.1	A.G. Kasan 3187				

OBSERVATIONS OF THE SATELLITE OF *NEPTUNE*,

By E. E. BARNARD.

The following observations of the satellite of *Neptune* are a continuation of those printed in the *Astronomical Journal*, Vol. XXVII, No. 638. The magnifying power employed throughout was 700 diameters. The observations are in Central Standard Time, or $6^h\ 0^m$ slow of Greenwich Mean Time.

The distance measures were made by the regular method of double distances (as in double star work), four measures being made in both positions of the movable wire. In this paper, to save space, the mean of these is given, instead of both sets.

1912–13	C.S. Time	P.A.	Dist.	Cps.	Remarks
	h m s	°	″		
Oct. 12	16 44 56	273.41	....	5	Satellite well seen,
	16 49 16	....	15.12	8	but seeing bad.
Oct. 15	16 47 8	90.11	....	5	Satellite faint.
	16 51 20	....	15.17	8	
Oct. 29	16 13 10	300.64	....	5	Satellite very f'nt
	16 18 6	....	15.83	8	from excessively bad seeing and moonlight.
Nov. 9	14 9 49	344.22	....	5	Very f'nt in clo'ds
	14 13 58	....	12.72	8	
Nov. 10	14 8 10	294.21	....	6	Excessively faint
	14 16 7	....	16.47	11	and difficult. Seeing excessively bad.
Nov. 16	12 28 16	291.88		6	Seeing poor but
	12 33 1		16.62	8	satellite well seen Mag. = 12.8.
Nov. 17	11 23 0	235.09		5	Satellite well seen.
	11 27 17		11.58	8	
Dec. 8	13 37 15	20.63		5	
	13 42 11		10.88	8	
Dec. 21	11 28 14	303.38	....	5	Well seen, 13
	11 33 16	....	16.32	8	mag. or brighter.
Dec. 22	11 42 20	260.30	....	5	
	11 47 18	....	14.00	8	
Dec. 28	13 17 25	246.99	....	5	Well seen.
	13 23 46	....	12.93	8	
Jan. 3	15 2 57	233.50	....	5	See'g exces'ly bad
	15 9 29	....	12.10	8	Satellite very difficult.
Jan. 4	11 37 51	158.80	....	5	Difficult. Seeing
	11 43 54	....	12.29	8	excessively bad.
Jan. 18	12 25 53	37.57	....	6	Excessively faint
	12 30 52	....	11.26	9	from bad seeing and thick sky.
Jan. 26	10 57 55	273.62	....	6	Excessively dif'it.
	11 5 34	....	15.31	11	Seeing very bad.
Jan. 28	10 3 42	133.43	....	6	Very f'nt in thick
	10 8 56	....	15.26	8	sky, but planet well defined.
Feb. 1	10 58 24	267.99	....	6	Fairly well seen.
	11 3 26	....	15.35	8	See'g fair to poor.
Feb. 4	8 27 37	88.99	....	6	Difficult. Seeing
	8 33 1	....	15.93	8	excessively bad.
Feb. 9	10 46 38	121.28	....	5	
	10 52 14	....	16.24	8	
Feb. 11	9 10 24	5.58	....	5	Faint from bad
	9 16 32	....	11.36	8	seeing.
Feb. 23	12 31 1	331.56	....	5	Very faint. Seeing
	12 40 52	....	12.73	10	excessively bad.

1912–13	C.S. Time	P.A.	Dist.	Cps.	Remarks
	h m s	°	″		
Mar. 4	8 38 11	151.61	....	5	Seeing fair.
	8 43 26	.....	13.15	8	
Mar. 9	10 49 18	220.08	....	5	13½ mag. Seeing
	10 54 54	.....	11.61	10	very good.
Mar. 11	10 3 58	97.64	....	6	Well seen but dim
	10 9 10	.....	16.47	8	from haze.
Mar. 15	11 5 34	208.78	....	5	Very faint and difficult.
	11 11 13	.....	11.47	6	Seeing exce'ly bad
Mar. 16	9 32 53	136.35	....	5	Very f'nt. Seeing
	9 38 42	.....	14.89	8	very bad. Bright moon near.
Mar. 29	9 21 7	84.05	....	6	Well seen See-
	9 26 2	...	14.92	8	ing fair.
Mar. 30	10 18 17	4.70	....	6	Faint. Sky poor,
	10 23 12	.....	11.00	10	seeing fair.
Apr. 6	9 29 59	296.44	.	7	Faint in clouds.
	9 39 28	.....	16.04	10	Only seen once in a while.
Apr. 13	9 22 19	244.45	.	7	Very faint in bright
	9 29 24	.....	12.84	10	moonlight. moon near.
Apr. 14	7 21 15	165.86	....	8	Faint. Seeing very
	7 28 33	....	11.85	8	poor. Bright moon near.

On March 29 a small double star was near and south of *Neptune*. The north and brighter component was measured with respect to *Neptune*.

9h 43m 30s	P. A.	199°.80	(3)
9 46 7	Dist.	114″.26	(3)

This gives the position of the double

1913.0 α 7h 39m 49s.4 δ + 20° 56′ 32″

Following are measures of the double:

Date			P.A.	Dist.	Mag.	Mag.	
			°	″			
1913.242	March	29	166.44	6.19	12	13½	
.244	"	30	169.73	6.13	12	13½	
.264	April	6	169.64	5.94			Faint in clouds.

The north component was measured with a 13th magnitude star south.

1913.264 April 6 152°.67 69″.76 Faint in clouds.

Yerkes Observatory, Williams Bay, Wisconsin, 1913 April 26.

OBSERVATIONS OF MINOR PLANETS,

MADE WITH THE 26-INCH AND THE 12-INCH EQUATORIALS AT THE U. S. NAVAL OBSERVATORY,

[Communicated by Captain J. L. Jayne, U. S. Navy, Superintendent].

Date	Wash. M.T.	*	Comp.	Δα	Δδ	App. α	App. δ	log p Δ α	log p Δ δ	Red. to App. Place α	Red. to App. Place δ
					(190) *Ismene*						
	h m s			m s	′ ″	h m s	° ′ ″			s	″
1910 Nov. 26	10 36 49	1	25 , 5	+1 58.64	+ 0 15.8	4 0 33.77	+12 39 32.5	9.105n	0.593	+3.42	+15.4*
					(569) *Misa*						
Dec. 8	11 56 3	2	30 , 6	+1 20.43	+10 27.8	5 15 21.95	+24 43 54.8	8.386n	0.328	+3.94	+10.0*
20	10 41 27	3	25 , 5	−1 51.65	+ 4 7.2	5 3 35.31	+24 21 48.1	8.767n	0.342	+4.08	+11.0*
21	8 41 36	4	25 , 5	+1 13.79	+ 3 34.7	5 2 44 57	+24 19 54.1	9.465n	0.433	+4.07	+11.4*
					(690) *Wratislavia*						
1911 Jan. 23	9 48 6	5	25 , 5	−1 21.13	− 0 33.4	8 12 25 77	+ 7 41 11.9	9 409n	0.67	+0.87	+ 2.4*
24	9 11 16	6	29 , 6	−0 23.82	+ 3 27.2	8 11 36.62	+ 7 42 26.8	9.488n	0.678	+0.88	+ 2.3*
25	8 51 4	7	28 , 6	−1 9.95	+ 3 49.3	8 10 46.92	+ 7 43 44.0	9.519n	0.682	+0.89	+ 2.2*
28	9 18 0	8	30 6	+1 18.53	− 1 57.8	8 8 16.71	+ 7 48 5.1	9.426n	0.671	+0.93	+ 2.0*
					(451) *Patientia*						
1912 Apr. 10	11 23 15	9	18 , 6	−0 18.92	− 3 30.6	12 36 43.98	+19 29 53.1	7.743	0.462	+1.94	− 9.9†
11	12 31 32	9	30 , 6	−1 5.59	− 2 7.2	12 35 57.31	+19 31 16.6	9.197	0.479	+1.94	− 9.8†
					(246) *Asporina*						
May 24	12 57 16	10	30,10	+0 34.91	+ 8 59.5	14 35 5.19	+ 8 26 26.0	9.454	0.667	+2.07	−11.7†
June 3	11 26 30	11	30,10	−0 53.56	− 0 17.5	14 30 23.62	+ 8 23 34.9	9.311	0.657	+2 06	−10.4†

Date Wash. M.T.	*	Comp.	Δα	Δδ	App. α	App. δ	log p Δ α	log p Δ δ	Red. to App. Place α	Red. to App. Place δ
					(491) *Carina*					
June 12 12ʰ 26ᵐ 24ˢ	12	30,10	+3ᵐ 39.05ˢ	+ 9′ 16.6″	16ʰ 42ᵐ 20.79ˢ	+ 3° 46′ 18.2″	9.131	0.704	+2.33	−10.8‡
17 12 1 56	13	24, 8	+2 20.41	− 1 1.5	16 38 55.85	+ 3 47 7.1	9.123	0.704	+2.39	−10.2‡
20 12 50 4	14	30,10	+2 3.55	− 6 58.3	16 36 57.17	+ 3 45 27.4	9.389	0.708	+2.40	−10.0‡
					(651) [1907 *AN*]					
Oct. 16 12 57 37	15	30,10	+1 15.98	+ 0 9.9	1 20 28.21	+ 2 51 54.3	9.188	0.713	+3.42	+24.0†
Nov. 2 10 32 48	16	25, 5	+0 55.42	+ 7 59.0	1 6 27.39	+ 2 45 6.2	8.461	0.713	+3.47	+23.6†
9 10 54 16	17	30, 6	−0 58.18	+ 0 58.6	1 1 50.44	+ 2 50 54.0	9.128	0.713	+3.43	+23.3‡
11 10 57 31	18	25, 5	−0 54.09	+ 5 0.4	1 0 42.70	+ 2 53 38.7	9.197	0.713	+3.43	+23.2‡
12 10 0 51	19	25, 5	+1 14.86	+ 7 45.7	1 0 12.00	+ 2 55 9.8	8.748	0.711	+3.43	+23.2‡
29 9 37 22	20	30,10	+0 37.36	− 3 8.6	0 54 50.31	+ 3 41 3.5	9.179	0.705	+3.34	+22.7†
					(115 *Thyra*					
Dec. 21 13 49 10	21	30, 6	−2 9.46	+ 0 13.1	6 0 58.86	+37 38 52.4	9.426	9.805	+5.51	+ 8.0†
22 11 34 34	22	30, 6	−1 31.74	− 4 8.4	5 59 48.27	+37 32 15.8	8.697*n*	9.297	+5.52	+ 8.3†
1913 Jan. 14 12 42 25	23	25, 5	−2 15.68	− 2 31.6	5 36 20.67	+33 56 26.1	9.554	0.219	+1.45	+11.3†
18 10 43 59	24	24, 8	−2 58.13	− 0 28.7	5 34 18.12	+33 16 6.5	9.161	9.987	+1.43	+11 3†
25 12 46 56	25	12, 4	−2 44.72	+ 1 4.1	5 32 16.97	+32 4 50.5	9.638	0.398	+1.40	+11.8†

Observers: *J. B. Eppes, †C. B. Watts, ‡H. E. Burton.

The observations of (190), (569), (690) and first one of (451) were made with the 12-inch equatorial; the second one of (451) and those of (246), (491), (651) and (115) were made with the 26-inch equatorial.

Mean Places of Comparison-Stars for the beginning of the year.

*	α	δ	Authority		*	α	δ	Authority	
1	3ʰ 58ᵐ 31.71ˢ	+12° 39′ 1.3″	A.G. Leipzig I	1187	14	16ʰ 34ᵐ 51.22ˢ	+ 3° 52′ 35.7″	A.G. Albany	5510
2	5 13 57.58	+24 33 17.0	A.G. Berlin B	1702	15	1 19 8.81	+ 2 51 20.4	A.G. Albany	381
3	5 5 22.88	+24 17 29.9	A.G. Berlin B	1664	16	1 5 28.50	+ 2 36 43.6	A.G. Albany	314
4	5 1 26.71	+24 16 8.0	A.G. Berlin B	1638	17	1 2 45.19	+ 2 49 32.1	A.G. Albany	297
5	8 13 46.03	+ 7 41 42.9	A.G. Leipzig II	4491	18	1 1 33.36	+ 2 48 15.1	A.G. Albany	290
6	8 11 59.56	+ 7 38 57.3	A.G. Leipzig II	4473	19	0 58 53.71	+ 2 47 0.9	A.G. Albany	275
7	8 11 55.98	+ 7 39 52.5	A.G. Leipzig II	4471	20	0 54 9.61	+ 3 43 49.4	A.G. Albany	245
8	8 6 57.25	+ 7 50 0.9	A.G. Leipzig II	4415	21	6 3 2.81	+37 38 31.3	A.G. Lund	3126
9	12 37 0.96	+19 33 33.6	A.G. Berlin A.	4727	22	6 1 14.49	+37 36 15.9	A.G. Lund	3104
10	14 34 28.21	+ 8 17 38.2	A.G. Leipzig II	6768	23	5 38 34.90	+33 58 46.4	A.G. Leiden	2250
11	14 31 15.12	+ 8 24 2.8	A.G. Leipzig II	6751	24	5 37 14.82	+33 16 23.9	A.G. Leiden	2235
12	16 38 39.41	+ 3 37 12.4	A.G. Albany	5533	25	5 35 0.29	+32 3 34.6	A.G. Leiden	2217
13	16 36 33.05	+ 3 48 18.8	A.G. Albany	5518					

SYSTEMATIC MOTION OF STARS OF TYPE G,

By BENJAMIN BOSS.

In *A.J.* 623-624 Prof. Lewis Boss found a considerable discordance between the apex of solar motion as determined from the stars of type G as compared with the position of the apex as derived from the treatment of stars of other types. Commenting upon this phenomenon he expressed the opinion that the difference is probably real.

Professor Campbell in *L. O. B.* 196 derived a smaller value for the solar motion as determined from the radial velocities of type G stars than the value obtained from the general solution, and was led to remark that the brighter stars of type G exhibited a tendency to partake of solar motion.

While we might attribute one peculiarity to chance, the double phenomenon seemed to call for investigation.

Two suggestions readily occur in explanation of the observed facts, either that there is a general drift motion of the G type stars with reference to the general system, or there might be one or more streams so directed as to cause the observed phenomena.

The material was at hand for a rough preliminary test of the nature of the disturbing motion by using the value of 19.5 km. for the solar velocity as determined by CAMPBELL, in the direction of A = 270°.7, D = + 34°.3 as determined by L. BOSS, with the values of 16.0 km., and 13.9 km., for the solar motion as derived by CAMPBELL from the G type stars, and the apex of solar motion as found for this type by L. BOSS at A = 259°.3, D = +42°.3. The results of the solutions indicated that the disturbing influence was, generally speaking, directed toward the antivertex of preferential motion, or toward what KAPTEYN, EDDINGTON and others call the apex of drift II.

For a more accurate determination of the apex of the drift or stream motion of the G type stars, the proper-motions of all the stars of that type were employed, as given in the *Preliminary General Catalogue*, with the exception of those motions which exceed 20″ per century. The latter were not used because they would most probably unduly influence the problem under the hypothesis adopted.

It seemed advisable to hypothetically free the observed proper-motions from solar motion. In order to accomplish this each observed motion was corrected for the value of the mean parallactic motion of 3″.85 due to the Sun's way through space, the solar apex being taken as A = 270.°7; D = +34°.3. The mean parallactic motion used was that derived by L. BOSS from his general solution of stars of all types, and the justification for its use lay in the fact that the mean parallactic value derived from the treatment of the G type stars alone would be spurious in case, as suspected, there exists any drift motion of this type.

Since the rough preliminary solutions indicated that the resultant effect, whatever it might be, was directed toward the antivertex of preferential motion, EDDINGTON's true apex of drift II, computed form the material of the *P.G.C.* was adopted as the approximate and preliminary position of the residual motion of the G type stars. Its position is given as A = 274°, D = −12° in *M.N.* Vol. LXXI, No. 1.

The deviations of the observed position angles of the stars, hypothetically corrected for the effects of solar motion, from the values computed on the assumption of a vanishing point at A = 274°; D = −12°, furnished the data for a differential solution to determine the apex of the group motion more thoroughly. The result of the solution gave the following differential corrections to the assumed preliminary position — dA = −1°.3; dD = +6°.8, placing the refined apex at

A = 272°.7 D = −5°.2

The agreement with EDDINGTON's true apex of drift II is very satisfactory, but the agreement with the antivertex of preferential motion as given by L. BOSS is almost perfect, the latter being given as A = 273°.8, D = −7°.0. It was considered unnecessary to repeat the solution for a more refined position of the apex as the differential corrections from the first solution were small, and the convergence toward the true value rapid from the form of solution. Likewise the material used did not warrant the employment of a greater degree of refinement. While the same argument would apply to an even greater degree to any attempted solution by separate areas, out of pure curiosity the celestial sphere was divided into four areas according to right ascension and separate solutions made. The results are as follows:

	A	D
h h	°	°
0− 6	247.9	− 2.6
6−12	297.2	+14.2
12−18	262.9	−27.8
18−24	280.0	− 9.0

As might have been expected the divergences are large, and their unsystematic character would lead us to believe that they are rather due to the sparsity of the material than to any real effect.

Considerable confidence should be manifested in the apex as derived from the general solution as a division of the hypothetical proper-motions in two groups 0° to 180° and 0° to −180° gives an even count, 225 in the one to 225 in the other.

The table accompanying this article shows the distribution of the differences in position-angles between the observed hypothetical proper-motions and their computed values on the hypothesis of a vanishing point at A = 272°.7, D = −5°.2. The tendency for motion in the given direction is strongly manifest in the preponderance of the small deviations from this direction of motion. There likewise appears to be a somewhat systematic character to the arrangement of this table, suggesting that the tendency of motion toward the antivertex varies in different portions of the sky, although under the hypothesis adopted a similar effect would be generated in case the mean distance of the stars in various regions differed to any extent. The investigation by HOUGH and HALM in *M.N.* Vol. LXX, No. 8 showed an apparent unequal distribution of the drift II stars. A comparison of their results with those in the table shows no points of similarity between the two, but it must be borne in mind that the method employed by HOUGH and HALM in differentiating between drifts I and II is open to criticism, as pointed out by EDDINGTON *M.N.* Vol. LXX, No. 8. On the other hand I am free to admit that the apparent systematic arrangement of the table may be solely due to the accidental distribution of the limited material treated.

The ratio of the preference of the G type stars for the direction of motion toward A = 272°.7, D = −5°.2, as compared with motion in the opposite direction is as 5 to 2.

Distribution of $\Delta\theta$.

R.A. \ $\Delta\theta$	0 to 30	30 to 60	60 to 90	90 to 120	120 to 150	150 to 180	−180 to −150	−150 to −120	−120 to −90	−90 to −60	−60 to −30	−30 to 0
h												
0	5	3	2	.	.	1	2	.	1	1	.	3
1	1	2	1	1	2	.	1	1	.	.	4	6
2	5	6	.	.	1	1	1	2	2		1	2
3	3	1	.	3	3	1	2		1	.	2	.
4	4	6	3	2	1	3	1	1		1	1	3
5	.		3	.	2	1	2	.			1	2
6	3	1	.	.	1	1	3	2	2	1	4	2
7	1	1	.	2	1	1	2	9	2	4	3	2
8	3	1	1	2	2	1	2	1	3	3	5	4
9	1	.	1	1	1	2	3	2	2	3	3	6
10	4	.	1		3	.	4	3	1	1	.	3
11	3	1	.	.	1	1		.	3	1	1	2
12	6	1	.	.	2	1	2	.	2		.	6
13	4	1	2	1	.	1	.	1	1		1	1
14	1	3	1	3	.	2	.	.	1	.	2	3
15	2	.	3	1	1	1	.	1	.	1	1	2
16	3	2	2	.	.	2	1	2		2	3	3
17	1	.	1		.	.	5	1	1	1	2	4
18	2	5	3	1	3	2	3		1	.	.	1
19	3	2	5	6	.	1	1	.	1	3		1
20	3	7	2	1	1	2	3	1	1	.		2
21	1	2	.	1	.	.		2	1	2		.
22	2	4	1	.	1	3		.	3	.	.	4
23	1	1	1		1	.	1	1	1	.	1	5

Feeling confident that many would object to the adoption of the value of 3″.85 for the correction of parallactic motion due to the Sun's motion through space, the proper-motions were also corrected for a value of the parallactic motion of 3″.12, which is the value derived by L. Boss in *A.J.*, 623-624, from the same G type stars employed in the present discussion. This value changed the ratio of the preference for the chosen direction from $\frac{271}{179}$ to $\frac{265}{185}$ or to all intents and purposes the former ratio of 3 to 2 was maintained. It was not considered necessary, therefore, to make a separate solution for the apex on the assumption of a mean parallactic motion of 3″.12, as the result could be only slightly altered in the direction of the antapex of solar motion.

Objection may also be raised to the exclusion of the large proper-motion stars of type G. However, the parallactic motion of these stars is so far greater than the values employed in the present discussion that a certain amount of distortion must necessarily result from their inclusion in the solution. Also the evidence points toward a proportionally greater number of stars that have abnormally large real motions among the stars of large proper-motion, so that they would not be representative of the general type. Nevertheless, it may be that their effect upon the solution should be considered. There is a larger proportion of these stars of type G moving in the direction of the vertex of preferential motion than those moving in the opposite direction. If their effect were to be considered in connection with the G type stars of lesser proper-motion, it would tend to more nearly equalize the distribution of the proper-motions.

An attempt to prove that the preferential motion of the G type stars is due to one or more streams of the *Taurus* stream type proved a failure.

Conclusions.

The stars of type G in so far as they are represented by the proper-motions of the *Preliminary General Catalogue*, and exclusive of those proper-motions amounting to over 20″.0 per century, exhibit a preference for motion toward the direction A = 272°.7; D = −5°.2, in the ratio of 3 to 2, though the inclusion of the larger proper-motion stars would tend to decrease this proportion. The effect, if real, indicates that as a class the G type stars are drifting toward the antivertex of preferential motion, though there is some probability that the addition of further data in the future will destroy this effect.

General Remarks.

The apparent physical stream here depicted must be differentiated from the streams of Kapteyn and Eddington which have no physical meaning but merely subserve the determination of the directions of the so-called preferential motion or other peculiarities of motion. The

streams of KAPTEYN and EDDINGTON include solar motion, but in the present case the effect of solar motion was first hypothetically eliminated so that the resultant motion is real, as far as indicated by the data used.

If real, such a bodily motion of a type of stars will influence the determination of solar motion, as evidenced by the value obtained for the apex of solar motion as derived from a solution of the same G type stars as used in the present discussion.

The fact that the solutions for the apex of solar motion by types yield results so closely similar, is a very fair indication that whatever may be the peculiarities of the distribution of motions among the various types, there must exist a certain similarity of symmetry in the distribution.

However, I would call attention to a peculiarity in the distribution of the apices of solar motion as derived from the solutions by type. With the exception of the G type, which as I have already stated is anomalous, these apices are strung along a parallel of approximately 21° galactic latitude. The parallelism to the plane of the galactic equator looks suspicious. Also some significance may attach itself to the close similarity between the A and F stars, and a similarity between the K and M stars. The apex of the B stars is, broadly speaking, located between the two sets, and if, as is probable, the B stars furnish us with a more reliable determination of the solar apex because of their distance from us and their small peculiar velocities, it would seem that in addition to the peculiarity of the G stars, there exists a tendency of motion in the early type stars as represented by the A and F types, opposite in resultant effect to a tendency of motion in the later types as represented by the K and M types. The data for the two pairs is as follows, the radial velocities being those given by CAMPBELL, *L.O.B.*, 196:

Type	A	D	V
	°	°	km
K	275.4	+40.3	−21.2
M	273.6	+38.8	−22.6
A	270.0	+28.3	−16.8
F	265.9	+28.7	−15.8

This matter will be studied later on.

Because the stars of type G, used in the general solution for solar apex by L. Boss, were relatively few, their exclusion changes the position but little. The general solution gives A = 270°.5, D° = +34°.3, while the exclusion of the G type stars places the solar apex at A = 271°.9; D = +32°.2.

I wish to acknowledge the assistance of Mr. HARRY RAYMOND who has offered me the benefit of his criticism and has personally performed much of the laborious computation.

SOME POSSIBLE STAR GROUPS,

By BENJAMIN BOSS.

There are a few cases of close similarity of proper-motion within a comparatively limited sky area, where the proper-motions are so small that it will require the confirmation of radial velocities to determine whether these groups represent a common motion or not. It might be well to call attention to a few of them.

In the accompanying tables the first column gives the number of the star in the *Preliminary General Catalogue.* The column headed μ indicates the full amount of the proper-motion, and the column θ gives the position-angles of the proper-motion with reference to the north pole. The type is taken from the Revised Harvard Photometry.

Dividing Group A into two sections and taking the brute mean of each there is a convergent indicated at roughly 10^h, −45°. That the motion is not entirely parallactic is indicated by the fact that the mean position angle of the group with reference to the apex of solar motion is 144°. The similarity of type would also lend confidence in the reality of the group.

In the case of Group B, the very small proper-motions would account for considerable latitude in the variation of the position-angles. This group is moving at right-angles to the direction of solar motion and very closely in the direction of the antivertex of preferential motion. The general similarity of type is again striking.

Group C has a small membership, but indicates a convergent at about $4^h\ 30^m$, −15°. It also is directed away from the antapex of solar motion.

GROUP A.

P.G.C. No.	R.A. 1875 h m s	Decl. 1875 ° ′	μ ″	θ °	Type
1313	5 19 53	+17 51	1.8	141.8	B_3
1315	20 8	+21 50	1.8	135.0	B_3
1333	24 36	+32 6	1.8	153.4	B_1
1335	24 53	+18 30	1.5	126.9	M*a*
1345	26 12	+18 27	1.3	116.6	A
1417	37 41	+16 2	1.8	139.4	.
1422	39 34	+15 46	1.6	129.8	A
1424	40 9	+17 41	1.1	153.4	A
1431	40 55	+20 49	2.7	145.7	A
1457	45 28	+27 35	2.3	142.1	A
1507	56 30	+20 8	1.6	142.4	B_2
1518	59 11	+23 39	1.9	137.1	.
1534	6 2 8	+23 8	2.1	127.4	A_2
1545	4 37	+19 49	1.6	150.3	A
1590	11 43	+23 31	1.3	141.3	. .
1612	17 2	+25 7	2.0	149.5	.

GROUP B.

P.G.C. No.	R.A. 1875 h m s	Decl. 1875 ° ′	μ ″	θ °	Type
2070	7 45 26	−46 4	1.1	221.2	B
2089	7 48 13	−38 32	1.4	197.1	B_3
2104	8 5 26	−48 19	1.7	197.4	A
2166	8 5 38	−46 59	1.4	192.1	B_3
2187	8 8 46	−35 31	1.3	161.6	B_3p
2217	8 16 38	−36 5	1.3	171.3	B_3
2267	25 15	−44 18	1.1	174.8	B
2297	32 8	−50 32	1.1	206.6	A
2324	36 29	−46 12	1.2	194.0	F_5p
2332	37 42	−44 58	0.8	203.2	B_5
2408	52 33	−52 15	0.9	180.0	B_5
2485	9 9 58	−37 5	1.2	166.0	G

GROUP C.

2606	9 37 29	−80 23	4.2	282.1	B
2715	10 11 28	−86 18	4.5	273.8	
2889	44 35	−79 53	4.7	267.6	B_5
3134	11 53 27	−77 32	4.3	249.4	B_5
3514	13 28 34	−55 3	4.5	221.4	A
4387	17 8 8	−80 44	4.2	178.6	Ma

GROUP D.

4679	18 22 53	−33 4	5.2	183.3	A_5
4732	35 12	−38 26	5.9	174.2	A
4734	35 57	−35 46	5.3	178.9	B_3
4784	47 31	−26 27	6.7	171.4	B_3
4955	19 19 2	−29 59	5.8	166.0	A
4973	22 8	−27 14	5.1	163.0	K_5
5108	51 36	−35 37	4.8	159.3	B_3

Group D is so nearly directed toward the antapex of solar motion that the entire effect may be parallactic in nature. It has been inserted, however, because on the chart representing the motions in this region, Group D stands out prominently.

There are other cases somewhat similar to Group D, but the reality of these groups is so doubtful that it is not considered worth while to include them. Groups A and B give special promise of sharing a common motion. In the case of Group B the radial velocity of star 2324 is given as +28.6 km., and there are two spectroscopic binaries, 2166 with an observed range +75.7 km. to −3.0 km., and 2408 with an observed range +67.5 km. to −3.7 km. There is great danger in mistaking parallactic motion for stream motion, so that when the position-angles of the proper-motion with reference to the apex of solar motion read near 180°, considerable caution should be exercised before claiming stream motion for such a group, even when the proper-motions are considerable. But in the cases cited, the motion is directed away from the antapex of solar motion, with the exception of the last group.

No attempt has been made to increase the membership of the groups given in the tables. The membership was governed by the apparent common motion as indicated on our charts of proper-motion and in each case is limited to the stars indicated on a chart governing a limited area. Adjoining areas might in some cases considerably increase the group membership.

NOTICE REGARDING THE DESIRABLE EXTENSION OF EPHEMERIDES.

C. D. PERRINE.

By the time many of the ephemerides of comets reach us through the usual channels of publication, they have almost or entirely expired. If computers will kindly take into account in the case of southern comets the six weeks which the publication and transmission require, it will be greatly appreciated by southern observers.

Cordoba, 1913 *March* 4.

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THE ORBITS OF FREELY FALLING BODIES,

By R. S. Woodward.

Historical Note.

It is now one hundred and ten years since the problem of the motion of a body falling freely from a considerable height above the earth's surface was first carefully considered by Gauss and Laplace. Two obvious sources of difficulty are presented by this problem, namely: first, the effect of the rotation of the Earth; and, secondly, the effect of the asymmetric distribution of the Earth's mass, for the fact that this mass is not centrobaric must be taken into account if any solution better than a rough approximation is to be attained. It should be said, however, that such a solution of the problem requires a knowledge of geodesy not available to Gauss and Laplace; for they are to be recalled as the most eminent founders of the theories of geodesy, while the necessary refined and elaborate observational data of this science have nearly all been acquired since their times. Thus, Gauss's investigation of the problem was published in March, 1803*; but this was ten years before Poisson's extension of Laplace's theorem in the theory of the potential function, twenty-five years before the appearance of the epochal paper by George Green (in which the term "potential function" first appeared) and thirty-seven years before the publication of the greatest of all papers on the potential function by Gauss himself. It is not surprising, therefore, that Gauss's treatment of the problem of falling bodies should now appear quite incomplete and unsatisfactory, although much more attention seems to have been given to it than to the far superior treatment furnished a little latter by Laplace. Concerning a critical point, which furnishes one of the chief reasons for the publication of the following paper, these eminent contemporaries were diametrically opposed. Gauss concludes that the falling body should undergo a small meridional deviation from the vertical towards the equator; but Laplace states quite positively that there is no such deviation†. Curiously enough, in spite of this discrepancy, nearly all writers, including authors of standard treatises, have reproduced the meridional deviation of Gauss, ignoring the much more important, though now quite inadequate, investigation of Laplace. To this, however, there is a noteworthy exception in the memoir** of Poisson published in 1838. In characteristic fashion Poisson investigates in a very comprehensive manner the flight of a projectile, of which a freely falling body presents a special case. In respect to meridional deviation he agrees with Laplace. Stated in his own words his conclusion is, "Il (le calcul) fait voir, en effet, que la déviation a lieu vers l'est, et qu'elle est nulle dans le sens du meridian."‡

But Poisson's object was rather a general theory of gunnery than the decision of a delicate question of geodesy, and he, like Gauss and Laplace, did not have access to the essential data; and, in anticipation of the sequel, it may be here remarked that there is still some doubt whether the data now available are adequate to a complete determination of this question, although we are now prepared to make a definite advance in that direction. In the meantime, while the theory of falling bodies has not improved appreciably since the date of Poisson's paper, experimentalists have done much to maintain interest in

*Fundamentalgleichungen für die Bewegung Schwerer Körper auf der rotirenden Erde. Werke, Band V.

†Gauss's paper appeared in a letter to Benzenberg in March, 1803. Laplace's paper was published in the following May, in the *Bulletin de la Société Philomatique.* In respect to the critical point in question he says "Je fais voir que, ... il ne doit point y avoir de déviation vers l'équateur." This paper was republished in the *Mécanique Céleste* as Chapitre V, Tome IV (edition of 1880). The title of this chapter is "De la chute des corps qui tombent d'une grande hauteur." In this he says "Je ferai voir, de plus, que sa déviation est nulle vers l'équateur."

**Mémoire sur le mouvement des projectiles dans l'air, en ayant égard à la rotation de la terre. Journal de *l'École Polytechnique,* Tome XVI, 1838.

‡It turns out, as shown in the later sections of this paper, that the meridional deviation is always away from the equator instead of towards it as Gauss and his followers have supposed. This deviation is a quantity of the second order and hence has failed to be evaluated. Gauss's expression for it is a mathematical fiction.

this classic problem, and more recently it has attracted the attention of the younger mathematicians in a way which seems to justify the hope that GAUSS and LAPLACE may have some worthy successors in the geodetic researches of the present century.*

OBJECT AND METHOD OF THIS PAPER.

In view of the opposing conclusions above referred to and in view of the uncertainties of experimentalists as to what would happen if it were possible to execute a series of precise experiments on bodies falling in vacuo from considerable heights, as well as in view of certain obvious defects in all discussions of the subject to be found in the standard memoirs and treatises, it has seemed worth while to undertake an independent investigation based on present day knowledge of geodesy. The defects in previous investigations which need to be remedied arise from three principal sources, namely: (*a*), inadequate expressions for the gravitational potential of the Earth for points outside its surface; (*b*), unjustifiable neglect of terms in the complete differential equations of motion of the falling body; (*c*), neglect of the distinction between geocentric and geographic (or astronomical) latitude. This last source of confusion and error has been very generally overlooked. Neglect of this distinction leads readily to false conclusions concerning meridional deviation, since this depends on and is of the same order as the difference between these two latitudes. In addition to carrying the order of approximation in the solution of the problem one step beyond that hitherto attained, it is incidentally desired to call attention to a pressing need in geodesy for a systematic gravimetric survey. The results of such a survey are required especially to determine the extent of inequality in the equatorial principal moments of inertia of the Earth, which play an important role in the problem of latitude variations as well as in the problem of falling bodies.

In deriving the equations of motion of the falling body the energy method of LAGRANGE is adopted, partly because it does not appear to have been used hitherto for this purpose and partly because it is peculiarly suited to the conditions of the problem. The equations of motion are expressed, however, in terms of three different systems of coordinates; so that the relative merits of these systems may be readily seen, while they afford the means essential for comparison of the results herein derived with those hitherto published. The route of approach to the problem here followed is that suggested by LAPLACE, but although his generalities are at once admirably clear and comprehensive we must reject his presentation of details as both obscure and inadequate. After the fashion of his day he entailed confusion and difficulty for the modern reader by making the radius of the Earth unity and by the unnecessary neglect of quantities without determining their relative magnitudes.

LAPLACE and POISSON in their investigations sought to take account of the resistance of the air, assuming it to be proportional to the square of the speed of the moving body. This was quite appropriate, and especially so in POISSON's work, since it was designed to supply the needs of gunnery. But present interest in the problem hinges mostly on the displacement of the falling body with respect to the vertical line through the point of departure, and this, as shown by POISSON in the memoir cited above, is but little affected by air resistance, since the orbit of the falling body is everywhere nearly normal to the equipotential surfaces in the atmosphere. For experiments in air, indeed, deviations due to unavoidable currents and to inequalities of surface friction even for the smoothest of spheres would be, in general, much greater than deviations due to normal air resistance. This latter, therefore, is left out of account in the following investigation.

GEOMETRY OF THE PROBLEM.

Let the position, P, of the body at any time, t, be referred to a system of rectangular coordinates having the Earth's center as origin, the axis of X directed toward the vernal equinox, the axis of Y lying in the plane of the Earth's equator 90° to the east of the equinox, and the axis of Z coincident with the Earth's axis of figure. The position of the body at any time t will then be given by the coordinates x, y, z. Let the general position of the body be defined also by a system of polar coordinates having the same origin, denoting the radius-vector of the body by r, its geocentric latitude by ψ and its longitude reckoned from the plane ZX towards the east by μ. The relations between these two systems are

$$\begin{aligned} x &= r \cos\psi \cos\mu \\ y &= r \cos\psi \sin\mu \\ z &= r \sin\psi \end{aligned} \qquad (1)$$

The coordinates of the initial position, P_0, say, of the body, or for the time $t = 0$, will be distinguished from the general coordinates in (1) by means of a zero suffix in any case.

*An interesting historical summary of the work of experimentalists on the deviation of falling bodies, along with a full account of his own carefully planned and executed experiments, is given by Professor E. H. HALL in Vol. I of Contributions from the Jefferson Physical Laboratory of Harvard University, pp. 179-254, 1903. The reader may also consult advantageously an elaborate historical and critical account of the experimental and mechanical features of the problem of falling bodies to be found in *La Rotation de La Terre, ses preuves mécaniques anciennes et nouvelles (avec deux appendices) par* J. G. HAGEN, *S. J., Roma*, 1911-1912.

Let the coordinates of the point P_1, say, vertically under the initial point (x_0, y_0, z_0) and in the tangent (horizontal) plane to which the body falls be distinguished by the numeral suffix unity, so that

$$\begin{aligned} x_1 &= r_1 \cos\psi_1 \cos\mu_1 \\ y_1 &= r_1 \cos\psi_1 \sin\mu_1 \\ z_1 &= r_1 \sin\psi_1 \end{aligned} \tag{2}$$

It is important to specify precisely how this point P_1 is located with reference to the initial point P_0. Imagine a basin of mercury at the point P_1. The surface of the mercury will be the level, or equipotential (or horizontal) surface through this point; and if it is located as here assumed the line joining the two points P_0 and P_1 will be normal to the surface of the mercury. In experiments of falling bodies the point P_1 is commonly located by suspending a plumb-bob from P_0 by means of a light cord or wire. But the optical method just suggested eliminates some questions which may arise as to the effect of the mass of the cord or wire as well as the more important questions of the effects of air currents on a plumb-bob thus suspended. It is essential to observe that the point P_1 located in the way just described will not be coincident with a point in the same plane determined by the line normal to the equipotential surface through P_0. In fact, since the equipotential surface at P_0 is more nearly spherical than the one at P_1 this latter point is nearer to the equator than the one in which the normal at P_0 pierces the plane tangent to the equipotential surface through P_1.

The most convenient of available forms for the potential of the Earth for points outside its surface is expressed in terms of polar coordinates referred to the principal axes of inertia and in terms of the principal moments of inertia of the Earth. Hence it is desirable to have equations (1) adapted to this form of the potential in question. To this end let L be the initial longitude of that equatorial principal axis of interia of the Earth which lies nearest to the west of the initial position of the falling body, and let λ denote the angle between the meridian plane through this principal axis and the meridian plane through the body at any time. Then, if ω denote the angular velocity of the Earth, the longitude of this principal axis after a time t will be $L + \omega t$, and the corresponding longitude of the body is $L + \omega t + \lambda$. Thus the longitudes which appear in equations (1) and (2) are, respectively,

$$\begin{aligned} \mu &= L + \omega t + \lambda \\ \mu_1 &= L + \omega t + \lambda_1 \end{aligned} \tag{3}$$

in which latter λ_1 and μ_1 refer to P_0 as well as to P_1, since both are in the same meridian plane.

Let the body be referred also to another system of coordinates ξ, η, ζ with origin at the point P_1, with the axis of ξ perpendicular to the normal at that point and directed positively in the meridian towards the equator; with axis of η perpendicular to the meridian and positive towards the east; and with the axis of ζ coincident with the normal and positive outwards. Denoting the direction cosines of these axes with respect to the axes x, y, z by

$$\begin{aligned} &\alpha_1,\ \beta_1,\ \gamma_1 \quad \text{for } \xi \\ &\alpha_2,\ \beta_2,\ \gamma_2 \quad \text{for } \eta \\ &\alpha_3,\ \beta_3,\ \gamma_3 \quad \text{for } \zeta \end{aligned}$$

the following relations hold:

$$\begin{aligned} \xi &= \alpha_1 (x - x_1) + \beta_1 (y - y_1) + \gamma_1 (z - z_1) \\ \eta &= \alpha_2 (x - x_1) + \beta_2 (y - y_1) + \gamma_2 (z - z_1) \\ \zeta &= \alpha_3 (x - x_1) + \beta_3 (y - y_1) + \gamma_3 (z - z_1) \end{aligned} \tag{4}$$

and conversely

$$\begin{aligned} x - x_1 &= \alpha_1\xi + \alpha_2\eta + \alpha_3\zeta \\ y - y_1 &= \beta_1\xi + \beta_2\eta + \beta_3\zeta \\ z - z_1 &= \gamma_1\xi + \gamma_2\eta + \gamma_3\zeta \end{aligned} \tag{5}$$

The values of the direction cosines which enter these equations are evident functions of the geographic latitude and the longitude of the point P_1. Thus, calling this geographic latitude φ, these cosines have the following values respectively:

$$\begin{aligned} \alpha_1 &= +\sin\varphi\cos\mu_1 & \alpha_2 &= -\sin\mu_1 \\ \beta_1 &= +\sin\varphi\sin\mu_1 & \beta_2 &= +\cos\mu_1 \\ \gamma_1 &= -\cos\varphi & \gamma_2 &= 0 \end{aligned}$$

$$\begin{aligned} \alpha_3 &= +\cos\varphi\cos\mu_1 \\ \beta_3 &= +\cos\varphi\sin\mu_1 \\ \gamma_3 &= +\sin\varphi \\ (\mu_1 &= L + \omega t + \lambda_1) \end{aligned} \tag{6}$$

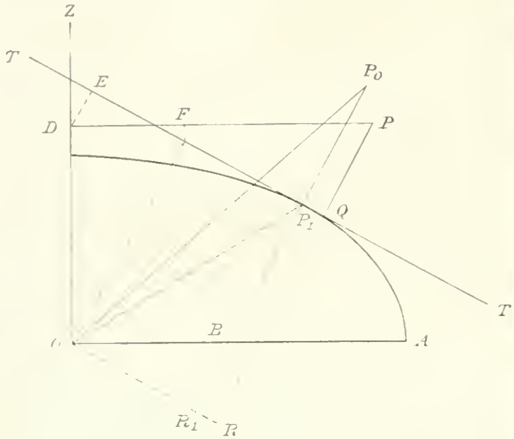
Frequent use will be made in the sequel of the differences between the initial and the subsequent values of the coordinates and especially of the polar coordinates in equations (1). Hence it is convenient to write

$$\begin{aligned} \Delta r &= r_0 - r \\ \Delta\psi &= \psi_0 - \psi \\ \Delta\lambda &= \mu - \mu_1 = \lambda - \lambda_1 \end{aligned} \tag{7}$$

Performing the substitutions specified in the second members of (4) and making use of the values in (6) and (7), there result the following exact relations between the variable coordinates

$$\begin{aligned} \xi &= r\sin(\varphi - \psi) - r_1 \sin(\varphi - \psi_1) \\ &\qquad - 2r\cos\psi\sin\varphi\sin^2\tfrac{1}{2}\Delta\lambda \\ \eta &= r\cos\psi\sin\Delta\lambda \\ \zeta &= r\cos(\varphi - \psi) - r_1\cos(\varphi - \psi_1) \\ &\qquad - 2r\cos\psi\cos\varphi\sin^2\tfrac{1}{2}\Delta\lambda \end{aligned} \tag{8}$$

$$
\begin{aligned}
r \cos \psi \cos \Delta\lambda &= r_1 \cos \psi_1 + \xi \sin \varphi + \zeta \cos \varphi \\
r \cos \psi \sin \Delta\lambda &= \eta \\
r \sin \psi \quad &= r_1 \sin \psi_1 - \xi \cos \varphi + \zeta \sin \varphi
\end{aligned}
\tag{9}
$$



The geometrical meanings of the terms in these equations are readily seen by reference to the adjacent diagram which shows the relations of the points and lines as they appear in, or when projected on, the meridian plane through the initial point P_0 and the origin P_1 for the coordinates ξ, η, ζ. Thus, in the first of (9) $r \cos \psi \cos \Delta\lambda$ is PD in the diagram, or the projection of the distance of P from the Earth's axis on the initial meridian plane through P_0 and P_1: and $r \cos \psi \cos \Delta\lambda \sin \varphi$ in the first of (8) is $\mathrm{PD} \sin \varphi = \mathrm{EQ}$. Also, $r \sin \psi \cos \varphi = \mathrm{OD} \cos \varphi = \mathrm{EF}$, and

$$
\begin{aligned}
r_1 \sin (\varphi - \psi_1) &= \mathrm{FP_1} : \text{so that} \\
\xi &= \mathrm{P_1 Q} = \mathrm{EQ} - \mathrm{EF} - \mathrm{FP_1}
\end{aligned}
$$

Similarly, observing that the second members of (9) are PD, η and OD respectively; making use of equations (1), (2), (5) and (6), and writing for brevity

$$
\begin{aligned}
\rho &= r_1 \cos \psi_1 + \xi \cos \varphi + \zeta \sin \varphi \\
\sigma &= r_1 \sin \psi_1 - \xi \sin \varphi + \zeta \cos \varphi \\
r^2 &= r_1^2 + \xi^2 + \eta^2 + \zeta^2 + 2 r_1 \xi \sin (\varphi - \psi_1) \\
&\qquad + 2 r_1 \zeta \cos (\varphi - \psi_1)
\end{aligned}
\tag{10}
$$

the following relations needed also in the sequel hold:

$$
\begin{aligned}
x &= \rho \cos \mu_1 - \eta \sin \mu_1 \\
y &= \rho \sin \mu_1 + \eta \cos \mu_1 \\
z &= \sigma
\end{aligned}
\tag{11}
$$

It is essential to observe that the geographic latitude φ is, like each of the geocentric latitudes ψ_0 and ψ_1, a constant in the problem here considered. The varying position of the body is completely specified by the coordinates r, ψ, μ or by ξ, η, ζ. Confusion concerning these facts and failure to recognize the importance of the quantities $(\varphi - \psi)$ and $(\varphi - \psi_1)$, small though they are, have led to general obscurity in the literature of this subject.

In addition to the preceding exact relations, all of which will be needed in establishing the equations of motion, several approximate expressions will be required. These may be conveniently derived also in this section.

Referring to the diagram let

$$
\delta_0 = \mathrm{P_1 P_0 O}, \quad \delta = \mathrm{P_0 O P_1}, \quad h = \mathrm{P_1 P_0} \tag{12}
$$

In accordance with previous designations

$$
\begin{array}{lll}
r_0 = \mathrm{OP_0} & r_1 = \mathrm{OP_1} & r = \mathrm{OP} \\
\psi_0 = \mathrm{P_0 O A} & \psi_1 = \mathrm{P_1 O A} & \varphi = \mathrm{P_1 B A}
\end{array}
$$

It is now proposed to express δ, δ_0, $(r_0 - r_1)$, $(r_0 - r)$, and ξ, η, ζ in terms of h, r_1, $(\varphi - \psi_1)$, Δr, $\Delta\psi$ and $\Delta\lambda$. To this end the triangle $\mathrm{P_0 OP_1}$, in the initial meridian plane gives

$$
\begin{aligned}
r_1 \sin \delta &= h \sin \delta_0 \\
r_0 &= r_1 \cos \delta + h \cos \delta_0 \\
\delta + \delta_0 &= \varphi - \psi_1
\end{aligned}
\tag{13}
$$

Writing for brevity

$$
\theta = \varphi - \psi_1, \qquad p = h/r_1 \tag{14}
$$

an application of MACLAURIN's series to the first of (13) gives

$$
\delta = p \sin \theta - \tfrac{1}{2} p^2 \sin 2\theta + \ldots \tag{15}
$$

Hence, to terms of the third order exclusive in θ and to terms of the second order inclusive in p there result

$$
\begin{aligned}
\delta &= \theta p (1 - p) \\
\delta_0 &= \theta (1 - p + p^2)
\end{aligned}
\tag{16}
$$

Similarly, the second of (13), by aid of (14) and (15), gives to terms of the second order inclusive in θ

$$
r_0 = r_1 + h - h (1 - p) \theta^2 \tag{17}
$$

Now, observing that

$$
\begin{aligned}
\psi &= \psi_0 - \Delta\psi = \psi_1 + \delta - \Delta\psi \\
\varphi - \psi &= \varphi - \psi_1 + \Delta\psi - \delta = \theta + \Delta\psi - \delta
\end{aligned}
$$

and that

$$
r = r_0 - \Delta r = r_1 + h - \Delta r
$$

to terms of the third order exclusive in h and θ, it follows that (8) become to terms of the second order inclusive in Δr, $\Delta\psi$, $\Delta\lambda$, h and θ

$$
\begin{aligned}
\xi = - r_1 \cos \theta (\delta - \Delta\psi) - (\Delta r - h) (\sin \theta - \delta + \Delta\psi) \\
- \tfrac{1}{2} r_1 \cos \psi_1 \sin \varphi (\Delta\lambda)^2
\end{aligned}
$$

$$\eta = r_1\,\Delta\lambda\,\{\cos\psi_1 - (\delta - \Delta\psi)\sin\psi_1\} - (\Delta r - h)\,\Delta\lambda\cos\psi_1 \tag{18}$$
$$\zeta = h - \Delta r + r_1\sin\theta\,(\delta - \Delta\psi) - \tfrac{1}{2}r_1\cos\psi_1\cos\varphi\,(\Delta\lambda)^2$$

The last of these equations shows that when the body reaches the plane $\xi\,\eta$, or when $\zeta = 0$, $(h - \Delta r)$ becomes a term of the second order. Hence for this case, to terms of the second order inclusive

$$\xi = -\,r_1(\delta - \Delta\psi) - \tfrac{1}{4}r_1(\Delta\lambda)^2\sin 2\psi_1$$
$$\eta = r_1\,\Delta\lambda\cos\psi_1\,\{1 - (\delta - \Delta\psi)\tan\psi_1\} \tag{19}$$
$$\Delta r = h + r_1\sin\theta\,(\delta - \Delta\psi) - \tfrac{1}{2}r_1\cos^2\psi_1\,(\Delta\lambda)^2$$

It is seen from the first of these equations that whether ξ is positive or negative depends on the factor $(\delta - \Delta\psi)$. In the following sections this factor is shown to be always positive. Hence the deviation ξ is always towards the adjacent pole in either hemisphere, or away from the equator instead of towards it as concluded by Gauss and his followers.

Equations of Motion of the Falling Body.

The mechanical system here considered consists of the Earth and the falling body, and this latter is assumed to be so near to the Earth that the motion in question is not sensibly affected by external masses, like those of the Moon and the Sun. Under this restiction the system is conservative; and in conformity with the Lagrangian method the equations of motion are derivable from the expressions for the kinetic and the potential energies of the system.

The falling body will be assumed to be centrobaric, so that its mass, which would appear as a factor common to all terms in the equations of motion, may be suppressed in these equations. The kinetic energy of the system is that of the rotating earth plus that of the falling body. But the first of these parts is an obvious constant. Hence, calling the mass of the body m, the kinetic energy of the system T, and denoting time-derivatives by accents it follows that

$$T = \text{constant} + \tfrac{1}{2}m\,(x'^2 + y'^2 + z'^2)\}$$

Making use of the relations (1) and (3) there results

$$T = \text{cons't} + \tfrac{1}{2}m\,\{r'^2 + r^2\psi'^2 + r^2(\omega + \lambda')^2\cos^2\psi\} \tag{20}$$

Similarly making use of the relations (3), (10) and (11)

$$\begin{aligned} T &= \text{cons't} + \tfrac{1}{2}m\,\{\xi'^2 + \eta'^2 + \zeta'^2 + 2\omega(\eta'\rho - \eta\rho')\} \\ &\quad + \tfrac{1}{2}m\,\omega^2(\eta^2 + \rho^2) \\ &= \text{cons't} + \tfrac{1}{2}m\,(\eta'^2 + \rho'^2 + \sigma'^2) \\ &\quad + m\,\omega\,(\eta'\rho - \eta\rho') \\ &\quad + \tfrac{1}{2}m\,\omega^2(\eta^2 + \rho^2) \end{aligned} \tag{21}$$

The potential energy of the system is that due to the attraction between the Earth's mass and the mass of the body. Denoting the potential per unit mass at any point external to the Earth by V, the distance from this point to any element of the Earth's mass by s, the volume of this element by dv, its density by ρ*, and the gravitation constant by k,

$$V = k\int\frac{\rho\,dv}{s} \tag{22}$$

The product Vm is the potential energy of the system; and the product of m by any one of the derivatives $\partial V/\partial r$, $\partial V/\partial\xi$, . . . gives the force (or moment in the case of an angular ordinate) in the direction r, ξ, ...

Now for any of the coordinates, r say, the Lagrangian equation of motion is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial r'}\right) - \frac{\partial T}{\partial r} = \frac{\partial V}{\partial r} \tag{23}$$

Applying this formula successively to the three coordinates r,ψ,λ of (20), and omitting the common factor m, there result

$$\begin{aligned} &\frac{d^2r}{dt^2} - r\left(\frac{d\psi}{dt}\right)^2 - r\left(\omega + \frac{d\lambda}{dt}\right)^2\cos^2\psi &&= \frac{\partial V}{\partial r} \\ &\frac{d}{dt}\left(r^2\frac{d\psi}{dt}\right) + r^2\left(\omega + \frac{d\lambda}{dt}\right)\cos\psi\sin\psi &&= \frac{\partial V}{\partial\psi} \\ &\frac{d}{dt}\left\{r^2\left(\omega + \frac{d\lambda}{dt}\right)\cos^2\psi\right\} &&= \frac{\partial V}{\partial\lambda} \end{aligned} \tag{24}$$

Similarly, applying (23) to the coordinates severally in (21), one finds

$$\begin{aligned} &\frac{d^2\xi}{dt^2} - 2\omega\sin\varphi\frac{d\eta}{dt} - \omega^2\rho\sin\varphi &&= \frac{\partial V}{\partial\xi} \\ &\frac{d^2\eta}{dt^2} + 2\omega\left(\sin\varphi\frac{d\xi}{dt} + \cos\varphi\frac{d\zeta}{dt}\right) - \omega^2\eta &&= \frac{\partial V}{\partial\eta} \\ &\frac{d^2\zeta}{dt^2} - 2\omega\cos\varphi\frac{d\eta}{dt} - \omega^2\rho\cos\varphi &&= \frac{\partial V}{\partial\zeta} \end{aligned} \tag{25}$$

$$\begin{aligned} &\frac{d^2\rho}{dt^2} - 2\omega\frac{d\eta}{dt} - \rho\omega^2 &&= \frac{\partial V}{\partial\rho} \\ &\frac{d^2\eta}{dt^2} + 2\omega\frac{d\rho}{dt} - \eta\omega^2 &&= \frac{\partial V}{\partial\eta} \\ &\frac{d^2\sigma}{dt^2} &&= \frac{\partial V}{\partial\sigma} \end{aligned} \tag{25$_1$}$$

These equations (24), (25) and (25_1) are exact. Laplace has derived the equivalents of (24) by the method of virtual displacements, but he neglects all terms in ω^2 and uses inadequate values for the second members. He puts $r = 1$ and draws no distinction between geocentric and geo-

* The symbol ρ used here temporarily to specify density should not be confounded with the same symbol used elsewhere in this paper to specify distance.

graphic latitudes. The equations of GAUSS, POISSON, and all subsequent writers, so far as I am aware, are equivalent in their first members to the first members of (25) if terms in ω^2 are neglected. All of these authors have used inadequate values for the second members. In general $\partial V/\partial\xi$ and $\partial V/\partial\eta$ are each replaced by zero, and $\partial V/\partial\zeta$ is replaced by the acceleration of gravity at the place of observation.*

It is important to observe, as already remarked, that the geographic latitude φ which is implied in (24) and which appears explicitly in (25) is constant. GAUSS was apparently aware of this fact, but neither he nor any of his followers seems to have considered it essential to draw a distinction between geographic and geocentric latitude.

In order to get adequate values for the second members of (24) and (25) it is essential to expand V in (22) in a series of spherical harmonics. The most appropriate available expansion is the following:**

$$V = \frac{Mk}{r} + \frac{k}{2r^3}\{C - \tfrac{1}{2}(B+A)\}(1-3\sin^2\psi) + \frac{3k}{4r^3}(B-A)\cos^2\psi\cos 2\lambda \tag{26}$$

In this M is the mass of the Earth; k is the gravitation constant; A, B, C, are the principal moments of inertia of the Earth in the order of increasing magnitude, A and B being the equatorial axial values and C that with respect to the Earth's axis of figure; and r, ψ, λ are the polar coordinates of equations (1) and (3).

For greater brevity and subsequent convenience write

$$\begin{aligned} \alpha &= Mk \\ \beta &= \{C - \tfrac{1}{2}(B+A)\}\,k \\ \gamma &= (B-A)\,k \end{aligned} \tag{27}$$

Using these abbreviations the derivatives of (26) with respect to r, ψ, λ are

$$\begin{aligned} \frac{\partial V}{\partial r} &= -\frac{\alpha}{r^2} - \frac{3\beta}{2r^4}(1-3\sin^2\psi) - \frac{9\gamma}{4r^4}\cos^2\psi\cos 2\lambda \\ \frac{\partial V}{\partial\psi} &= -\frac{3\beta}{r^3}\cos\psi\sin\psi - \frac{3\gamma}{r^3}\sin 2\psi\cos 2\lambda \\ \frac{\partial V}{\partial\lambda} &= -\frac{3\gamma}{2r^3}\cos^2\psi\sin 2\lambda \end{aligned} \tag{28}$$

* See, for example, APPEL, *Traité de Mécanique Rationnelle*, Tome II, p. 278, edition of 1904; or ROUTH, *Advanced Rigid Dynamics*, pp. 25-30, 5th edition, 1892.

** See, for example, *Die Mathematischen und Physikalischen Theorien der Höheren Geodasie* VON F. R. HELMERT, Teil II, p. 72; or TISSERAND, *Traité de Mécanique Céleste*, Tome II, p. 363.

GEODETIC DATA ESSENTIAL.

Before passing on to the integration of the equations (24) and (25) it is desirable to know the relative magnitudes of the constants α, β, γ and thereby of the quantities in the second members of (28). Other geodetic data will also be needed in the sequel, and hence all of them may be assembled in this section.

To this end let

a = major semi axis of Earth's ellipsoid of revolution.
b = minor semi axis of same.
e = eccentricity of this ellipsoid, so that $a^2e^2 = a^2-b^2$.
v = volume of Earth
ρ_m = mean density of Earth

Then the first of equations (27) becomes

$$\alpha = Mk = v\,(k\,\rho_m), = \tfrac{4}{3}\pi\,a^2 b\,(k\,\rho_m) \tag{29}$$

The volume v in this equation can be computed with a high degree of precision from the known dimensions of the Earth. The product of the mean density of the Earth by the gravitation constant is known with a precision indicated by five significant figures. This product is, in fact, as shown by me in the *Astronomical Journal*, No. 424, 1897,

$$k\,\rho_m = 36797 \times 10^{-11}/(\mathrm{sec})^2 \tag{30}$$

with a probable error less than one unit in the fifth significant figure.†

Since the time of LAPLACE the equatorial moments of inertia of the Earth have been generally assumed to be equal. They are probably nearly so, but existing knowledge does not enable us to state with precision how nearly. If their difference were as great as one-half of one per cent of either it would account completely for the prolongation of the Eulerian cycle disclosed in the variation of latitudes;‡ so that one may say that this difference lies between the limits zero and one two hundredth of A or B. But in the absence of exact information, which can be gained apparently only by means of a comprehensive gravimetric survey of the Earth, the prevailing assumption that $A = B$ will be adopted here.

† This product may be written

$$k\rho_m = \frac{3\pi}{\tau^2}$$

wherein τ is the periodic time of an infinitesimal satellite which would revolve about the Earth, just grazing the equator, if there were no atmospheric or other resistance; and the value of τ is $1^h\,24^m\,20.9^s$. Thus the theory of the paper referred to shows how the regularity of the speed of rotation of the Earth might be checked by means of gravimetric surveys made from time to time if the attainable precision of such surveys were high enough.

‡ See the author's paper "Mechanical Interpretation of Variations of Latitudes," *Astronomical Journal*, No. 345, 1895.

In conformity with this assumption the most trustworthy available values of the moments of inertia in question appear to be those derived by HARKNESS in his "Solar Parallax and Related Constants."* By a least square adjustment involving many quantities, the most of which are from necessity not very closely interrelated, he computed the following values:

$$\begin{aligned} A = B &= 0.325029\ Ma^2 \\ C &= 0.326094\ Ma^2 \\ C - \tfrac{1}{2}(B + A) &= 0.001065\ Ma^2 \end{aligned} \tag{31}$$

Adopting these values the constant γ in (27) and (28) is zero and the second member of the third of equations (24) vanishes. The meaning of this latter condition is that the orbit of the falling body does not depend on its longitude.

Using CLARKE's spheroid of 1866 and the C. G. S. system of units the following values of the above constants and their logarithms will be needed:

$$\begin{array}{lll} a = 637825900 & 8.8047021 & \\ b = 635663500 & 8.8032273 & \\ e^2 = 0.00676866 & 7.8305030 - 10 & (32) \\ v = 1083221 \times 10^{21} & 27.0347201 & \end{array}$$

The relation between the geographic latitude φ of any place on the Earth's surface and the corresponding geocentric latitude ψ, to which frequent reference has already been made, is defined exactly by the equation.**

$$\tan \psi = (1 - e^2) \tan \varphi$$

whence if

$$q = \frac{e^2}{2 - e^2}$$

$$\varphi - \psi = q \sin \varphi - \tfrac{1}{2} q^2 \sin 2\varphi + \ldots \tag{33}$$

$$(\log q = 7.53095 - 10)$$

There will be needed also a convenient formula for computing the radius vector R, say, of any point on the Earth's surface. A special value of R will be r_1 which figures in equations (2), (8), (14), etc. Such a formula for the natural logarithm of R is†

$$\begin{aligned} \log R = \log \frac{a\,(2 - e^2)}{1 + \sqrt{1 - e^2}} & \\ + (q - q_1) \cos 2\varphi & \\ - \tfrac{1}{2}(q^2 - q_1^2) \cos 4\varphi & \\ + \ldots\ . & \end{aligned}$$

wherein q has the value given by the first of (33) and

$$q_1 = \frac{1 - \sqrt{1 - e^2}}{1 + \sqrt{1 - e^2}}$$

Applying the values of the constants given by (32) and introducting the modulus of common logarithms in the series of cosines of multiples of φ there results for the adopted spheroid and for the centimeter as unit of length

$$\begin{aligned} \log R = 8.8039666 & \\ + [3.86769] \cos 2\varphi & \\ - [1.2737] \cos 4\varphi & \\ + \ldots & \end{aligned} \tag{34}$$

wherein the numbers in square brackets are logarithms of the coefficients of $\cos 2\varphi$ and $\cos 4\varphi$, respectively, for units of the seventh decimal place. Thus, for example, if $\varphi = 0$, $\log R = \log a = 8.8039666 + 7373.8 - 18.8 = 8.8047021$ as given in (32)

Making use of the values in (29) to (33)

$$\begin{aligned} \log \alpha &= 20.6005325‡ \\ \log \beta &= 35.23728 \end{aligned} \tag{35}$$

There will be needed likewise in the sequel the angular velocity of the Earth, ω, and ω^2, along with the quantity $3\beta/R^5$, or its equivalent $3\beta/r^5$. This latter quantity appears in the integration of (24) in connection with ω^2 and it is essential to know that they are of the same order as well as of the same kind.

Since the mean solar second is the unit of time in the C. G. S. system

$$\omega = \frac{2\pi}{86164.1}$$

Hence

$$\begin{aligned} \log \omega &= 5.86285 - 10 \\ \log \omega^2 &= 1.72571 - 10 \end{aligned} \tag{36}$$

To compute a value of $3\beta/R^5$ for the present purposes of comparison it will suffice to use the value of $\log R$ for mid-latitude $\varphi = 45°$, since the value of R changes very slowly with latitude. Thus by (34) $\log R = 8.8039685$ for $\varphi = 45°$; and by reference to (35) it is found that

$$\log\,(3\beta/R^5) = 1.69456 - 10 \tag{37}$$

INTEGRATION OF EQUATIONS (24).

Since the second member of the third of equations (24) is assumed to be zero, it expresses constancy of moment of momentum of the falling body with respect to the Earth's axis of figure and is immediately integrable. Us-

* Washington, Government Printing Office, 1891.

** See HELMERT, *Geodäsie*, Teil I, p. 60.

† See CHAUVENET's *Spherical and Practical Astronomy*, Vol. I, p. 100, edition of 1871.

‡ The data for a do not justify a precision indicated by 7-place logarithms, but this number of places is here used in order to secure accurate numerical verification in the results furnished by the integrals of equations (24) and (25).

ing accents for brevity to indicate time derivatives, and observing that the initial angular velocity of the body is ω the first integral of this equation is

$$r^2(\omega + \lambda')\cos^2\psi = \text{constant}$$
$$= r_0^2\,\omega\cos^2\psi_0,$$

whence, making use of equations (7) and (17),

$$\omega + \lambda' = \omega\left(1 + 2\frac{\Delta r}{r_1} - 2\,\Delta\psi\tan\psi_1\right)$$
$$\lambda' = 2\omega\left(\frac{\Delta r}{r_1} - \Delta\psi\tan\psi_1\right) \tag{38}$$

to terms of the second order inclusive in $(\Delta r/r_1)$, $\Delta\psi$ and ω.

Introducing the first of (38) in the second of (24) and making use again of (7) and (17) there results to terms of the second order inclusive

$$\frac{d}{dt}\left\{\left(1 - \frac{\Delta r - h}{r_1}\right)(\Delta\psi)'\right\} = \frac{1}{2}\left(\omega^2 + \frac{3\beta}{r_1^5}\right)\sin 2\psi_1$$

whence, to the same order of approximation,

$$(\Delta\psi)' = \frac{1}{2}\left(\omega^2 + \frac{3\beta}{r_1^5}\right)t\sin 2\psi_1$$
$$\Delta\psi = \frac{1}{4}\left(\omega^2 + \frac{3\beta}{r_1^5}\right)t^2\sin 2\psi_1 \tag{39}$$

Referring now to the first of equations (24) it is seen from the first of (39) that the term in $\psi'^2 = (\Delta\psi)'^2$ is of the fourth order in ω^2 and its equivalent magnitude $3\beta/r_1^5$. Hence, neglecting this term and making use of the relations (7) and (17) it follows that to terms of the second order inclusive

$$(\Delta r - h)'' = n^2(\Delta r - h) + g$$

wherein

$$n^2 = \frac{2\alpha}{r_1^3} + \frac{6\beta}{r_1^5}(1 - 3\sin^2\psi_1) - 3\omega^2\cos^2\psi_1$$
$$g = \frac{\alpha}{r_1^2} + \frac{3\beta}{2r_1^4}(1 - 3\sin^2\psi_1) - (r_1 + 4h)\,\omega^2\cos^2\psi_1 \tag{40}$$

It should be observed that while the quantity here denoted by g is analogous to, it is not identical with the so-called "acceleration of gravity," or, more exactly, the resultant acceleration due to the attraction and to the rotation of the Earth.

The integral of the last differential equation is

$$n^2(\Delta r - h) = D\cosh(nt) + E\sinh(nt) - g$$

in which D and E are constants. Since

$$\Delta r - h = -h \text{ and } (\Delta r - h)' = 0$$

for $t = 0$, these constants are

$$D = g - hn^2$$
$$E = 0$$

Thus

$$\Delta r = \frac{g - hn^2}{n^2}\left\{\cosh(nt) - 1\right\} \tag{41}$$

Expanding the hyperbolic cosine this equation gives

$$\Delta r = \tfrac{1}{2}gt^2\left\{1 + \tfrac{1}{12}(nt)^2 + \ldots\right\}$$
$$- \tfrac{1}{2}h(nt)^2\left\{1 + \tfrac{1}{12}(nt)^2 + \ldots\right\} \tag{42}$$

Recurring to the second of (38) it now becomes by means of (41) to terms of the second order inclusive

$$n^2 r_1\lambda' = 2\omega(g - hn^2)\left\{\cosh(nt) - 1\right\}$$

whence, since $\lambda = \lambda_0$ for $t = 0$,

$$n^3 r_1\Delta\lambda = 2\omega(g - hn^2)\left\{\sinh(nt) - nt\right\} \tag{43}$$

Expanding the hyperbolic sine in this expression there results

$$r_1\Delta\lambda = \tfrac{1}{3}g\omega t^3\left(1 - \frac{hn^2}{g}\right)\left\{1 + \tfrac{1}{20}(nt)^2 + \ldots\right\} \tag{44}$$

This along with δ of (15) and $\Delta\psi$ of (40) inserted in the second of (19) will give the easterly deviation η.

Since, as shown below, δ is of the same order as $\Delta\psi$, and since the latter by (39) is of the second order, the value of the easterly deviation to this order of approximation is

$$\eta = r_1\Delta\lambda\cos\psi_1 \tag{44$_1$}$$

wherein $r_1\Delta\lambda$ is given by (44).

It is now essential to clearness of ideas to know the numerical values of the various terms which enter the equations (39) to (44). To this end consider the case for which $\varphi = 45°$. For this case the factors in the second members of (39) are

$$\tfrac{1}{4}\omega^2 = 13294 \times 10^{-13}$$
$$(3\beta/4r_1^5) = 12374 \times 10^{-13}$$

Similarly, the terms in equations (40) are

$$\begin{aligned}
(2\alpha/r_1^3) &= +\ 30878 \times 10^{-10}\\
\frac{6\beta}{r_1^5}(1 - 3\sin^2\psi_1) &= -\quad\ \ 49 \times 10^{-10}\\
-3\omega^2\cos^2\psi_1 &= -\quad\ \ 80 \times 10^{-10}\\
\hline
n^2 &= \quad\ 30749 \times 10^{-10}
\end{aligned}$$

$$\begin{aligned}
(\alpha/r_1^2) &= +\ 983.095 \text{ cm.}/(\text{sec})^2\\
\frac{3\beta}{2r_1^4}(1 - 3\sin^2\psi_1) &= -\quad 0.788 \text{ cm.}/(\text{sec})^2\\
-(r_1 + 4h)\,\omega^2\cos^2\psi_1 &= -\quad 1.693 \text{ cm.}/(\text{sec})^2\\
\hline
g &= \quad 980.614
\end{aligned}$$

Hence in equations (42) and (44) for the latitude 45°

$$(nt)^2 = 30749 \times 10^{-10} \text{ for } t = 1^s$$
$$= 30749 \times 10^{-8} \text{ for } t = 10^s$$

Hence for $t = 10$ seconds or less, or for falls of 490 meters or less, the terms in (nt) of (42) and (44) are relatively unimportant. But their precise values are easily computed. Thus if $t = 10^s$ (42) gives

$$\begin{aligned} \tfrac{1}{2}gt^2 &= 49030.70 \text{ cm.} \\ \tfrac{1}{24}gt^2\,(nt)^2 &= +\ 1.26 \\ -\ \tfrac{1}{2}h\,(nt)^2 &= -\ 7.54 \\ \Delta r &= 49024.42 \end{aligned}$$

For the same latitude and for $t = 10$ sec. (44) and (44_1) give

$$\begin{aligned} \tfrac{1}{3}g\omega t^3 \cos 45° &= \quad 16.855 \text{ cm.} \\ -\ \tfrac{1}{3}g\omega t^3\left(\frac{hn^2}{g}\right)\cos 45° &= -\ 0.003 \\ \eta &= \quad 16.852 \end{aligned}$$

Suppose now, for the sake of further numerical illustration, the data given are $\varphi = 45°$ and the total time of fall is 10^s. In any experiments on falling bodies it would be practicable to measure the height h, of P_0 above P_1, directly. But it may be here shown how h may be computed. The values just derived give by aid of (42)

$$\Delta r = 49024.42 \text{ cm.}$$

This is a sufficient approximation to h to permit computation of δ. Thus, since $\log r_1 = 8.8039685$ and since $p = h/r_1$, $\log p = 5.88668 - 10$. Also, by the second of (33), $\log(\varphi - \psi_1) = \log \theta = 7.53095 - 10$. Hence the first three terms of (15) give

$$\delta = p\theta - p^2\theta - \tfrac{1}{6}p\theta^3$$

$$\begin{aligned} p\theta &= 26146 \times 10^{-11} \\ p^2\theta &= \quad\ \ 2 \times 10^{-11} \\ \tfrac{1}{6}p\theta^3 &= \quad\ \ 5 \times 10^{-13} \\ \hline \delta &= 26144 \times 10^{-11} \end{aligned}$$

Recurring to the value $\Delta\psi$ computed above and making $t = 10$ seconds it is seen that for this value of t

$$\begin{aligned} \delta - \Delta\psi &= (26144 - 25668)\ 10^{-11} \\ &= +\ 476 \times 10^{-11} \end{aligned}$$

Putting $\zeta = 0$ in the last of (18) it is seen that

$$h = \Delta r - r_1\theta\,(\delta - \Delta\psi) + \tfrac{1}{2}r_1 \cos^2\psi_1\,(\Delta\lambda)^2$$

and by (43)

$$r_1\,(\Delta\lambda)^2 = \frac{(g\omega t^3)^2}{9r_1}$$

Thus

$$h = 49024.42 - 0.011 + 223 \times 10^{-9}$$

that is, for this case, h and Δr differ by only one-tenth of a millimeter, or in round numbers by one part in 5,000,000.

Finally, the values of ξ and η given by the first two respectively of (19) are

$$\begin{aligned} \xi &= -\ \ 3.03 \text{ centimeters} \\ \eta &= +\ 16.85 \text{ centimeters} \end{aligned}$$

It is thus seen that for a fall of 490.24 meters, in vacuo, the northerly deviation in the northern hemisphere and the southerly deviation in the southern hemisphere would be not only an easily appreciable quantity in itself but a considerable fraction of the easterly deviation.

For the sake of ready reference the numerical data and results derived in this section may be here collected in compact form.

$\varphi = 45°$	$t = 10$ seconds
$\log r_1 = 8.80397$	$\tfrac{1}{4}\omega^2 = 13294 \times 10^{-13}$
$\log \omega = 5.86285 - 10$	$(3\beta/4r_1^5) = 12374 \times 10^{-13}$
$\log p = 5.88668 - 10$	$\delta = 26144 \times 10^{-11}$
$\log \theta = 7.53095 - 10$	$\Delta\psi = 25668 \times 10^{-11}$
$\log g = 2.99149$	$\delta - \Delta\psi = 476 \times 10^{-11}$

$$\begin{aligned} \Delta r &= 49024.43 \text{ cm} \\ h &= 49024.42 \\ \xi &= -\ \ 3.03 \\ \eta &= +\ 16.85 \end{aligned}$$

Integration of Equations (25) and (25_1).

In order to complete this investigation it is essential to integrate the equations (25) and (25_1) and to show that the values they give for the coordinates are equivalent to the values given by (8), (18) and (19). But quite independently of the confirmation thus afforded of the conclusions already reached it appears to be worth while to show that equations (25) and (25_1) are suspectible to no less precise and complete treatment than equations (24).

Observing the relations (10) and using accents to indicate derivatives with respect to the time, equations (25) and (25_1) give

$$\begin{aligned} \rho'' - 2\omega\eta' - \omega^2\rho &= \frac{\partial V}{\partial \xi}\sin\varphi + \frac{\partial V}{\partial \zeta}\cos\varphi = \frac{\partial V}{\partial \rho} \\ \eta'' + 2\omega\rho' - \omega^2\eta &= \frac{\partial V}{\partial \eta} \\ \sigma'' \qquad\qquad &= -\frac{\partial V}{\partial \xi}\cos\varphi + \frac{\partial V}{\partial \zeta}\sin\varphi = \frac{\partial V}{\partial \sigma} \end{aligned} \qquad (45)$$

The most direct way to get the derivatives of V which appear in the second and third members of these equations is to express V as a function of ξ, η, ζ and r. Thus by

means of (9), (10) and (27) equation (26) becomes

$$V = \frac{\alpha}{r} - \frac{\beta}{r^3} + \frac{3\beta}{2r^5}(\eta^2 + \rho^2) \tag{46}$$

This gives by reference to the first of (10)

$$\begin{aligned} \frac{\partial V}{\partial \xi} &= \frac{\partial V}{\partial r}\frac{\partial r}{\partial \xi} + \frac{3\beta\rho \sin \varphi}{r^5} \\ \frac{\partial V}{\partial \eta} &= \frac{\partial V}{\partial r}\frac{\partial r}{\partial \eta} + \frac{3\beta\eta}{r^5} \\ \frac{\partial V}{\partial \zeta} &= \frac{\partial V}{\partial r}\frac{\partial r}{\partial \zeta} + \frac{3\beta\rho \cos \varphi}{r^5} \end{aligned} \tag{47}$$

The last of equations (10) gives

$$\begin{aligned} \frac{\partial r}{\partial \xi} &= \frac{1}{r}(\xi + r_1 \sin \theta) \\ \frac{\partial r}{\partial \eta} &= \frac{\eta}{r} \\ \frac{\partial r}{\partial \zeta} &= \frac{1}{r}(\zeta + r_1 \cos \theta) \end{aligned} \tag{48}$$

Bearing in mind that $\theta = \varphi - \psi_1$, (47) and (48) give for the second members of (45) the following values respectively:*

$$\begin{aligned} \left(\frac{\partial V}{\partial r} + \frac{3\beta}{r^4}\right)\frac{\rho}{r} &= \frac{\partial V}{\partial \rho} \\ \left(\frac{\partial V}{\partial r} + \frac{3\beta}{r^4}\right)\frac{\eta}{r} &= \frac{\partial V}{\partial \eta} \\ \frac{\partial V}{\partial r}\frac{\sigma}{r} &= \frac{\partial V}{\partial \sigma} \end{aligned} \tag{49}$$

Now for brevity write

$$\begin{aligned} \nu_1^2 &= -\left(\frac{\partial V}{\partial r} + \frac{3\beta}{r^4}\right)\frac{1}{r} - \omega^2 \\ &= \frac{\alpha}{r^3} - \frac{6\beta}{r^5} + \frac{15\beta \cos^2 \psi}{2r^5} - \omega^2 \\ \nu_2^2 &= -\frac{\partial V}{r\partial r} \\ &= \frac{\alpha}{r^3} - \frac{3\beta}{r^5} + \frac{15\beta \cos^2 \psi}{2r^5} \end{aligned} \tag{50}$$

Then, making use of these in (49), equations (45) become

$$\begin{aligned} \rho'' - 2\omega\eta' + \nu_1^2\rho &= 0 \\ \eta'' + 2\omega\rho' + \nu_1^2\eta &= 0 \\ \sigma'' + \nu_2^2\sigma &= 0 \end{aligned} \tag{51}$$

*These equations (49) may be derived also, but not so readily, from (26) and (28). It should be noticed, however, that $\partial V/\partial r$ in the first of (28) is not the same as $\partial V/\partial r$ derived from (46).

The quantities ν_1 and ν_2 which oppear in these equations are evidently angular velocities, as may be seen likewise by reference to (35) and (50). They are also seen to be variable, since they depend on ξ, η, ζ through the radius vector r and through the quantity $(\eta^2 + \rho^2)$. But the extent of this variation is small as shown by the last of equations (10), which gives to terms of the second order inclusive in ξ, η, ζ

$$r = r_1 + \xi \sin \theta + \zeta \cos \theta + \frac{1}{2r_1}(\xi \cos \theta + \zeta \sin \theta)^2 + \frac{\eta^2}{2r_1}$$

For such applications as are here considered, for which $t = 10$ seconds or less, the maximum value of $(r - r_1)$ in this last equation amounts to only 1/13000th part of r_1. The effect of this variation on ξ, η, ζ for falls of 490 meters or less will therefore be small, since the equations (51) are exact except for the approximations involved in ν_1 and ν_2. For the calculation of these quantities, however, one may use the time-average value of r and thus secure a higher order of precision in the calculation of ξ, η, ξ than is justified by the data from which the quantities α and β of (27) are derived. This time-average value of r is from (43)

$$\begin{aligned} r &= r_1 + h - \frac{g}{2t}\int_0^t t^2 dt \\ &= r_1 + h - \tfrac{1}{3}h. \end{aligned}$$

Using this value of the radius vector the parts which enter into, and the values of, ν_1^2 and ν_2^2 are for the mid latitude $\varphi = 45°$ as follow:

$$\begin{aligned} \log r &= 8.8039908 \\ \frac{\alpha}{r^3} &= 15436.90 \times 10^{-10} \\ \frac{6\beta}{r^5} &= 98.96 \times 10^{-10} \\ \frac{15\beta \cos^2 \psi}{2r^5} &= 61.85 \times 10^{-10} \\ \nu_1^2 + \omega^2 &= 15399.79 \times 10^{-10} \\ \omega^2 &= 53.17 \times 10^{-10} \\ \nu_1^2 &= 15346.62 \times 10^{-10} \\ \nu_2^2 &= 15449.27 \times 10^{-10} \end{aligned}$$

Returning to equations (51) in which ν_1 and ν_2 are now supposed constant, multiply the first by ρ' and the second by η', add products and integrate. The result is

$$\rho'^2 + \eta'^2 + (\rho^2 + \eta^2)\,\nu_1^2 = \text{constant}^{**}$$

** This equation means that the kinetic energy

$$\tfrac{1}{2}m\,(\rho'^2 + \eta'^2) + \tfrac{1}{2}m\,(\rho^2 + \eta^2)\,\nu_1^2 = \tfrac{1}{2}m\rho_0^2\nu_1^2$$

is conserved in the motion of the body.

This first integral and the equations from which it is derived are satisfied by

$$\rho = a_1 \cos (n_1 t + \epsilon_1) + a_2 \sin (n_2 t + \epsilon_2)$$
$$\eta = b_1 \sin (n_1 t + \epsilon_1) + b_2 \cos (n_2 t + \epsilon_2)$$

in which a_1, a_2, b_1, b_2, n_1, n_2, ϵ_1 and ϵ_2 are constants, provided that the first six of these constants conform to the following conditions:

$$\begin{aligned} a_1\, a_2\, \nu_1^2 - b_1\, b_2\, n_1\, n_2 &= 0 \\ b_1\, b_2\, \nu_1^2 - a_1\, a_2\, n_1\, n_2 &= 0 \\ a_1\, (\nu_1^2 - n_1^2) - 2\; b_1\, n_1\, \omega &= 0 \\ a_2\, (\nu_1^2 - n_2^2) + 2\; b_2\, n_2\, \omega &= 0 \\ b_1\, (\nu_1^2 - n_1^2) - 2\; a_1\, n_1\, \omega &= 0 \\ b_2\, (\nu_1^2 - n_2^2) + 2\; a_2\, n_2\, \omega &= 0 \end{aligned}$$

These conditions require first that

$$b_1 = a_1, \qquad b_2 = a_2 \tag{i}$$

secondly, that

$$n_1\, n_2 = \nu_1^2 \tag{ii}$$

and thirdly, that n_1 and n_2 be the roots of the quadratic

$$n^2 \pm 2n\omega = \nu_1^2 \tag{iii}$$

or that

$$\begin{aligned} n_1 &= \pm \sqrt{\nu_1^2 + \omega^2} - \omega \\ n_2 &= \pm \sqrt{\nu_1^2 + \omega^2} + \omega^* \end{aligned} \tag{iv}$$

Introducing the relations of (i) in the above equations for ρ and η they become

$$\begin{aligned} \rho &= a_1 \cos (n_1 t + \epsilon_1) + a_2 \sin (n_2 t + \epsilon_2) \\ \eta &= a_1 \sin (n_1 t + \epsilon_1) + a_2 \cos (n_2 t + \epsilon_2) \end{aligned} \tag{52}$$

The constants in these equations are determined by the conditions that for $t = 0$,

$$\begin{aligned} \rho &= \rho_0 = r_1 \cos \psi_1 + h \cos \varphi \\ \rho' &= 0 \\ \eta &= 0 \\ \eta' &= 0 \end{aligned}$$

These conditions require that

$$\epsilon_1 = 0, \text{ or a multiple of } \pi,$$

$$\epsilon_2 = \frac{\pi}{2}, \text{ or an odd multiple of } \frac{\pi}{2},$$

$$\begin{aligned} a_1 &= \rho_0 \frac{n_2}{n_1 + n_2}, \quad a_2 = \rho_0 \frac{n_1}{n_1 + n_2}, \\ a_1 + a_2 &= \rho_0, \qquad a_1 - a_2 = \rho_0 \frac{n_2 - n_1}{n_1 + n_2}, \end{aligned} \tag{53}$$

Now write for brevity

*Both sets of values of n_1 and n_2 are applicable; that is, the positive pair and the negative pair of roots satisfy the conditions and give the same results for η and ρ.

$$w^2 = \nu_1^2 + \omega^2 \tag{54}$$

Then the values in (iv) become

$$\begin{aligned} n_1 &= w - \omega, \quad \text{or } -(w + \omega) \\ n_2 &= w + \omega, \quad \text{or } -(w - \omega) \end{aligned} \tag{55}$$

Making use of the relations (53) to (55) in (52) the latter become

$$\begin{aligned} \rho &= \rho_0 \cos wt \cos \omega t + \rho_0 \left(\frac{\omega}{w}\right) \sin wt \sin \omega t \\ \eta &= \rho_0 \left(\frac{\omega}{w}\right) \sin wt \cos \omega t - \rho_0 \cos wt \sin \omega t \end{aligned} \tag{56}$$

Recurring to the last of the equations (51) its complete integral is seen to be

$$\sigma = a_3 \cos (\nu_2 t + \epsilon_3)$$

in which a_3 and ϵ_3 are constants. They are determined by the conditions that for $t = 0$

$$\begin{aligned} \sigma &= \sigma_0 = r_1 \sin \psi_1 + h \sin \varphi = a_3 \cos \epsilon_3 \\ \sigma' &= -a_3\, \nu_2 \sin \epsilon_3 = 0 \end{aligned}$$

These give $\epsilon_3 = 0$ and $a_3 = \sigma_0$. Hence the value sought is

$$\sigma = \sigma_0 \cos \nu_2 t \tag{57}$$

This last equation and (56) are the complete integrals of equations (51) and it remains only to derive ξ and ζ from (57) and the first of (56). For this purpose write

$$\begin{aligned} 1 - \tau_1 &= \cos wt \cos \omega t + \frac{\omega}{w} \sin wt \sin \omega t \\ &= 1 - \{\sin^2 \tfrac{1}{2} (w + \omega)\, t + \sin^2 \tfrac{1}{2} (w - \omega) t\} \\ &\quad + \frac{\omega}{w} \{\sin^2 \tfrac{1}{2} (w + \omega) t - \sin^2 \tfrac{1}{2} (w - \omega) t\} \\ 1 - \tau_2 &= \cos \nu_2 t = 1 - 2 \sin^2 \tfrac{1}{2} \nu_2 t \end{aligned} \tag{58}$$

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Then observing the values of ρ and σ given by (10) it follows from (57) and the first of (56) that

$$\begin{aligned} \zeta \cos \varphi + \xi \sin \varphi &= \rho_0 (1 - \tau_1) - r_1 \cos \psi_1 \\ \zeta \sin \varphi - \xi \cos \varphi &= \sigma_0 (1 - \tau_2) - r_1 \sin \psi_1 \end{aligned}$$

Introducing the values of ρ_0 and σ_0 these equations give

$$\begin{aligned} \zeta &= h - \xi \tan \varphi - (r_1 + h + r_1 \sin \theta \tan \varphi - 2 r_1 \sin^2 \tfrac{1}{2}\theta) \tau_1 \\ \zeta &= h + \xi \cot \varphi - (r_1 + h - r_1 \sin \theta \cot \varphi - 2 r_1 \sin^2 \tfrac{1}{2}\theta) \tau_2 \end{aligned} \tag{59}$$

The forms given to the last equations lend themselves readily to precise computations of ξ and ζ, especially the latter when ξ is derived independently. By elimination from (59) there result

$$\begin{aligned} \xi &= -r_1 \tau_1 \sin \theta + \tfrac{1}{2} (r_1 + h) (\tau_2 - \tau_1) \sin 2\varphi \\ &\quad - 2 r_1 (\tau_2 - \tau_1) \cos \varphi \cos (\varphi - \tfrac{1}{2}\theta) \sin \tfrac{1}{2}\theta \\ \zeta &= h - (r_1 + h - 2 r_1 \sin^2 \tfrac{1}{2}\theta)\, \tau_1 \\ &\quad - (r_1 + h) (\tau_2 - \tau_1) \sin^2 \varphi \\ &\quad + 2 r_1 (\tau_2 - \tau_1) \sin \varphi \cos (\varphi - \tfrac{1}{2}\theta) \sin \tfrac{1}{2}\theta \end{aligned} \tag{60}$$

For the easterly deviation the second of (56) gives

$$\eta = \tfrac{1}{2}\rho_0\left(1+\frac{\omega}{w}\right)\sin\,(w-\omega)t - \tfrac{1}{2}\rho_0\left(1-\frac{\omega}{w}\right)\sin\,(w+\omega)t \qquad (61)$$

The expressions (60) and (61) are exact except for the small uncertainties in the angular velocities ν_1 and ν_2 (and hence in the functions τ_1 and τ_2) to which reference has already been made. But these expressions are so different in form from those previously derived in this paper for the coordinates ξ, η, ζ that the equivalence of the several sets of values is not immediately evident. To remove this obscurity, observe from (58) that to terms of the fourth order exclusive

$$\tau_1 = \tfrac{1}{2}(\nu_1 t)^2, \quad \tau_2 = \tfrac{1}{2}(\nu_2 t)^2$$

and by (50)

$$(\tau_2-\tau_1) = \tfrac{1}{2}(\nu_2^2-\nu_1^2)\,t^2 = \tfrac{1}{2}\left(\omega^2+\frac{3\beta}{r_1^5}\right)t^2$$

Also by (40) and (50) it is seen that to the first order of approximation

$$\begin{aligned} r_1\,\tau_1 \sin\,\theta &= \tfrac{1}{2}r_1 \sin\,\theta\,(\nu_1 t)^2 \\ &= \tfrac{1}{2}gt^2 \sin\,\theta \\ &= h \sin\,\theta \end{aligned}$$

and this by (14) and (15) to the same order of approximation is $r_1\delta$. Thus it appears that the first two terms in the second member of the first of (60) are identical with the first two terms in the second member of the first of (19).

Similarly, (61) gives to terms of the third order inclusive

$$\begin{aligned} \eta &= \tfrac{1}{3}\frac{\rho_0}{r_1}(r_1\,\nu_1^2)\,\omega t^3 \\ &= \tfrac{1}{3}g\omega t^3 \cos\,\psi_1 \end{aligned}$$

which agrees with (44_1) to the first order of approximation. It is sufficiently evident therefore that the expressions for ξ, η, ζ derived in this section are interconvertible with the corresponding expressions derived in previous sections of this paper. A more precise test of numerical equivalence and correctness of all the formulas derived is afforded, however, by the computation of ξ and ζ from the equations (59) and (60). This computation is somewhat tedious but the essential details of it are given below. In order to secure a high degree of precision it is necessary to have some of the data, already given above, expressed by seven significant figures and to use 7-place logarithms in some of the calculations. Adopting the same values for the latitude and the time of fall t as hitherto, namely, 45° and 10 seconds, respectively, it is first required to compute τ_1 and τ_2 from (58). For this purpose it is found that

$$\begin{aligned} w &= 0.001240959 \\ \omega &= 0.000072920 \\ w+\omega &= 0.001313879 \\ w-\omega &= 0.001168039 \\ \log\,\nu_2 &= 7.0944540-10 \end{aligned}$$

Hence follow

$$\begin{aligned} \sin^2\tfrac{1}{2}(w+\omega)t &= 0.00004315630 \\ \sin^2\tfrac{1}{2}(w-\omega)t &= 0.00003410748 \\ \tau_2 = 2\sin^2\tfrac{1}{2}\nu_2 t &= 0.00007724536 \\ \tau_1 &= 0.00007673198 \\ \tau_2-\tau_1 &= 0.00000051338 \end{aligned}$$

With these values and with

$$\log r_1 = 8.8039685, \qquad \log \sin\,\theta = 7.53095-10$$

it is now easy to get the separate terms of (59) and (60). Thus, assuming that the value of ξ derived for the same case in the previous section is available, and making $\zeta = 0$, these two equations should give the same value for h. It will be observed that a first approximation to h is $r_1\,\tau_1$ or $r_1\,\tau_2$ and that the mean of these two values is a still closer approximation to h. By the first of (59) the results are

$$\begin{aligned} r_1\,\tau_1 &= 48859.03 \text{ cm.} \\ h\,\tau_1 &= +\quad 3.76 \\ r_1\,\tau_1 \sin\,\theta &= +165.92 \\ 2r_1\,\tau_1 \sin^2\tfrac{1}{2}\theta &= -\quad 0.28 \\ +\,\xi &= -\quad 3.03 \\ \hline \text{Sum} = h &= 49025.40 \end{aligned}$$

By the second of (59) the results are

$$\begin{aligned} r_1\,\tau_2 &= 49185.93 \\ h\tau_2 &= +\quad 3.79 \\ r_1\,\tau_2 \sin\,\theta &= -167.03 \\ 2r_1\,\tau_2 \sin^2\tfrac{1}{2}\theta &= -\quad 0.28 \\ -\,\xi &= +\quad 3.03 \\ \hline \text{Sum} = h &= 49025.44 \end{aligned}$$

Similarly, the first of equations (60) gives for ξ

$$\begin{aligned} -\,r_1\,\tau_1 \sin\,\theta &= -\ 165.92 \text{ cm.} \\ +\,\tfrac{1}{2}\,(r_1+h)\,(\tau_2-\tau_1) \sin 2\varphi &= +\ 163.46 \\ -\,2r_1\,(\tau_2-\tau_1) \cos^2\varphi \sin\tfrac{1}{2}\theta &= -\quad 0.56 \\ \hline \text{Sum} = \xi &= -\quad 3.02 \end{aligned}$$

And the second of (60) gives for h (ζ being zero for the case under consideration)

$$
\begin{aligned}
(r_1 + h)\,\tau_1 &= \,48862.79 \text{ cm.}\\
-2r_1\,\tau_1 \sin^2 \tfrac{1}{2}\theta &= - \quad 0.28\\
(r_1 + h)\,(\tau_2 - \tau_1) \sin^2 \varphi &= + \,163.46\\
-2r_1\,(\tau_2 - \tau_1) \sin\varphi \cos\varphi \sin\tfrac{1}{2}\theta &= - \quad 0.56\\
\text{Sum} = h &= \,49025.41
\end{aligned}
$$

The difference between the value of $\xi = -3.02$ centimeters here derived and that furnished by the preceding section is one-tenth of a millimeter and is no greater than should be expected from the different processes of calculation. The range amongst the different values of h here computed is less than one-millionth part of it, showing that the computations by (59) and (60) are consistent with one another. If the value of ξ from the first (60) is used in (59) the range in h will be reduced by one-half. But a more searching numerical test of the correctness and of the high order of approximation of the formulas derived is furnished by a comparison of the value of h coming from the second of (60) with that furnished by the last of (18), or by (42). Thus, from equation (42) $h = 49024.41$cm. and by equation (60) $h = 49025.41$, showing a difference of 1.00 centimeter, or a discrepancy in round numbers of one part in 50,000*. This must be regarded as quite satisfactory in view of the different processes required in deriving the formulas (42) and (60). This degree of accordance is, indeed, as already indicated, much higher than the geodetic data require, for existing knowledge of them does not justify the use of more than five significant figures in such computations as are here made.

Summary.

For the benefit especially of the non-mathematical reader the salient features of this investigation may be briefly summarized as follows:

(1) The problem of falling bodies has hitherto been inadequately treated by reason of neglect of the effect of the square of the angular velocity of the Earth and other terms of the same order, and by equal neglect of the effect of the difference between the geocentric latitude of the point of departure of the body and the geographical latitude of the horizontal plane to which the body falls. Failure to consider these effects has led to erroneous conclusions in respect to the meridional deviation of the body, the real deviation being, in fact, opposite to that derived by Gauss and most later investigators.

(2) The problem is here investigated anew in the light of geodetic data not available to Gauss and Laplace who were the pioneers in this field. The motion of the body is referred to four sets of interconvertible coordinates and the equations of motion for each of three of these sets are derived without neglect of any of the factors involved. Each set of equations of motion is integrated so as to include all terms of the second order in the angular velocity of the Earth and in equivalent factors. The values thus derived independently for the coordinate displacements of the body from the three sets of integrals are shown to be consistent with each other and to give a degree of numerical accordance surpassing that justified by the precision of the geodetic data on which calculations are based.

(3) The displacement of the body parallel to the meridian plane through its initial position is shown to be always away from the equator, or towards the adjacent pole in either hemisphere, and the amount of this displacement is a very sensible fraction of the easterly deviation.

(4) If the equatorial principal moments of inertia of the Earth are equal, as here assumed, the deviation of the body does not depend on its longitude. Attention is called to the need of further observational data to determine the approximation of equality of these moments of inertia. Such data would be supplied by an elaborate gravimetric survey of the Earth.

(5) The equality of the equatorial principal moments of inertia just referred to requires constancy of moment of momentum of the body in respect to the Earth's axis of figure. The same condition requires that the part of the kinetic energy of the body due to its motion parallel to the equator remain constant.

(6) The last set of the equations of motion referred to gives the matually orthogonal distances (ρ, η, σ) of the body from the Earth's axis of figure, from the meridian plane through the initial position and from the plane of the equator, respectively. It is remarkable that the last of these distances, σ, is to a high degree of approximation a simple harmonic function whose amplitude is the initial value of σ; while the other two distances (ρ, η) are expressed with equal approximation by sums respectively of two simple harmonic functions of two different angles.

*If in computing the angular velocities ν_1 and ν_2 the value r_1 of the radius vector had been used in place of the value $r_1 + \frac{2}{3}h$ the above noted discrepancy would have been about one part in 16,000.

SUNSPOT OBSERVATIONS,

MADE AT BERWYN, PENN., WITH A 4½-INCH REFRACTOR,

By ALDEN W. QUIMBY.

1913	Time	New Grs.	Total Grs.	Total Spots	Fac. Grs.	Def.
Jan. 1	8	–	1	3	–	fair
2	8	–	1	1	–	fair
3	8	–	1	1	1	fair
4	11	–	–	–	1	fair
5	8	–	–	–	–	fair
6	8	–	–	–	–	fair
7	9	–	–	–	–	v. poor
8	8	–	–	–	–	fair
9	8	–	–	–	–	fair
10	10	–	–	–	–	poor
11	2	–	–	–	–	fair
13	8	–	–	–	–	fair
14	8	–	–	–	–	fair
15	8	–	–	–	–	fair
16	12	–	–	–	–	poor
17	8	–	–	–	–	fair
18	4	–	–	–	–	poor
19	8	–	–	–	–	poor
20	9	–	–	–	–	poor
21	8	–	–	–	–	poor
22	8	–	–	–	–	poor
23	2	–	–	–	–	v. poor
24	8	–	–	–	–	poor
25	8	–	–	–	–	fair
26	8	1	1	2	–	fair
28	8	–	–	–	–	fair
29	11	–	–	–	–	poor
30	8	–	–	–	–	fair
31	8	–	–	–	–	fair
Feb. 1	8	–	–	–	–	fair
2	8	–	–	–	–	fair
4	8	–	–	–	–	fair
5	8	–	–	–	–	fair
6	8	–	–	–	–	fair
7	8	–	–	–	–	fair
8	8	–	–	–	–	fair
9	8	–	–	–	–	fair
10	8	–	–	–	–	fair
12	8	–	–	–	–	fair
13	8	–	–	–	–	fair
14	8	–	–	–	–	fair
15	8	–	–	–	–	fair
16	8	–	–	–	–	fair
17	8	–	–	–	–	fair
18	8	–	–	–	–	fair
19	8		–	–	–	fair
20	8		–	–	–	fair
21	9	1	1	6	–	fair
22	4	–	1	4	–	fair
23	2	–	1	2	–	poor
24	9		1	2		poor
25	8		1	2		fair
26	8	–	1	2	–	fair
28	8	–	–	–	1	fair
Mar. 1	9				–	v. poor
2	8	–	–	–	–	fair
3	8	–	–	–	–	fair
4	8	–	–	–	–	fair

1913	Time	New Grs.	Total Grs.	Total Spots	Fac. Grs.	Def.
Mar. 5	4	–	–	–	–	fair
6	2	–	–	–	–	fair
7	8	–	–	–	–	fair
8	8	–	–	–	–	fair
9	8	–	–	–	–	fair
10	8	–	–	–	–	fair
11	8	–	–	–	–	fair
12	8	–	–	–	–	fair
13	9	–	–	–	–	v. poor
14	4	–	–	–	–	v. poor
16	8	–	–	–	–	fair
17	8	–	–	–	–	fair
18	8	–	–	–		fair
19	8	–	–	–	–	fair
20	8	–	–	–	–	fair
21	8	–	–	–	–	fair
22	8	–	–	–	–	fair
23	8	–	–	–	–	fair
24	9	–	–	–	–	poor
25	4	–	–	–	–	fair
26	8	–	–	–	–	poor
27	8	–	–	–	–	fair
28	8	–	–	–	–	fair
29	8	–	–	–	–	fair
30	8	–	–	–	–	fair
31	8	–	–	–	–	fair
Apr. 1	8	–	–	–	–	fair
2	7	–	–	–	–	fair
3	3	–	–	–	–	fair
4	7	–	–	–	–	fair
5	7	1	1	2	–	fair
6	7	–	1	2	–	fair
7	7	–	1	1	–	fair
8	7	–	–	–	–	fair
9	7	–	–	–	–	fair
10	7	–	–	–	–	fair
11	11	–	–	–	–	poor
14	9	–	–	–	–	poor
15	8	–		–	–	good
16	10	–	–	–	–	poor
17	7	–	–	–	–	fair
18	7	–	–	–	–	good
19	8	–	–	–	–	good
20	8	–	–	–	–	good
21	7	–	–	–	–	fair
22	7	–	–	–	–	fair
23	7	–	–	–	–	fair
24	7	–	–	–	–	fair
25	7	–	–	–	–	fair
26	7	–	–	–	–	fair
28	7	–	–	–	–	fair
29	2	–	–	–	–	fair
30	7	–	–	–	–	fair
May 1	7	–	–	–	–	fair
2	7	–	–	–	–	fair
3	7	–	–	–	–	fair
4	7	–	–	–	–	fair
5	5	–	–	–	–	fair

1913	Time	New Grs.	Total Grs.	Total Spots	Fac. Grs.	Def.
May 6	7	–	–	–	–	fair
7	7	–	–	–	–	fair
8	7	–	–	–	–	fair
9	7	–	–	–	–	fair
10	7	–	–	–	–	fair
11	7	–	–	–	–	fair
12	7	–	–	–	–	fair
13	7	–	–	–	–	fair
14	7	–	–	–	–	fair
15	7	–			–	fair
16	7	–	–	–	–	fair
18	7	–	–	–	–	fair
19	7	–	–	–	–	fair
20	7	–	–	–	–	fair
21	7	–	–	–	–	fair
22	10	–	–	–	–	fair
23	7	–	–	–	–	fair
24	7	–	–	–	–	fair
25	7	–	–	–	–	fair
26	7	–	–	–	–	fair
27	9	–	–	–	–	fair
28	7	–	–	–	–	fair
29	7	–	–	–	–	fair
30	7	–	–	–	–	fair
31	7	–	–	–	–	fair
June 1	7	–	–	–	–	fair
2	7	–	–	–	–	fair
3	7	–	–	–	–	fair
4	3	–	–	–	–	fair
4	7	–	–	–	–	fair
5	7	–	–	–	–	fair
6	7	–	–	–	–	fair
7	7	–	–	–	–	fair
8	7		–	–	–	fair
9	7	–	–	–	–	fair
10	7	–	–	–	–	fair
11	7	–	–	–	–	fair
12	7	–	–	–	–	fair
13	7	–	–	–	–	fair
14	7	–	–	–	–	fair
15	7	–	–	–	–	fair
16	7	–	–	–	–	fair
17	7	–	–	–	–	fair
18	7	–	–	–	–	fair
19	7	–	–	–	–	fair
20	7	–	–	–	–	fair
21	7	–	–	–	–	fair
22	7	–	–	–	–	fair
23	7	–	–	–	–	poor
24	7	–	–	–	–	fair
25	7	–	–	–	–	fair
26	7	–	–	–	–	fair
27	2	–	–	–	–	fair
28	7	–	–	–	–	fair
29	7	–	–	–	–	fair
30	7	–	–	–	–	fair

NOTE ON THE PROPER MOTION OF WEISS-ARGELANDER 7113,

By J. G. PORTER.

The proper motion assigned to WEISS-ARGELANDER 7113 in the preface of the Washington A. G. Zones is $+0^s.026$ in right ascension and zero in declination, a correction of $-20''$ being applied to ARGELANDER's declination.

Two recent observations at this observatory by Dr. SMITH show, however, that the difference in declination is also due to proper motion in that coordinate, and the star falls in the class of those with yearly motion greater than half a second of arc.

The data for the star, reduced to N. F. K. system, are as follows:

		R.A. 1900.0	Decl. 1900.0
		h m s	° ′ ″
W. Arg.	1851.0	8 36 9.60	−15 58 6.8
Bord.	94.2	10.76	32.7
Wash. A. G.	94.8	10.73	33.4
Ci.	1912.7	11.22	41.5
		P.M. $+0^s.026$	$-0''.54$

Cincinnati Observatory, July 9, 1913.

ATMOSPHERIC DISTURBANCES AND VARIABLE REFRACTION,

By C. D. PERRINE.

Several references have appeared recently on some phase of this subject. To what has been said I may add the principal results of my own experience covering a period of fifteen years. Briefly it is as follows. In 1898 I began work on the parallax of a few of the planetary nebulæ with the 36-inch Lick refractor, the measures being made visually with the filar micrometer. After taking every precaution and investigating many possible sources of error I came to the conclusion that the chief trouble was in refraction effects which caused the images to be displaced bodily and that the mean of ten or even twenty "settings" did not remove the discordances in many cases. The discordances were about as large in declination as right ascension showing that they were not due to errors of driving of the telescope. A consideration of the size of the spurious disc of the star showed that these discordances could not be due to simple accidental errors in the bisection of the image, but that there must have been actual relative displacements of the images. The distances measured varied up to 3′. (In the measurement of double stars where such a high degree of accuracy is shown, the distances are very small).

I then gave up visual measurement for parallax purposes.

My next active investigations in this line were during the *Eros* campaign in 1900. I wished to have an independent check upon the scale-value of the photographs of *Eros* taken for parallax purposes with the Crossley reflector. For that purpose I designed, and had constructed, a device in which a shutter was operated electrically by a standard clock to make short exposures of a sufficiently bright star every two seconds.

These images showed clearly the same bodily displacements as had the visual images with the refractor, and of about the same magnitude. Trails were also obtained which showed similar displacements of the image. Many of the displacements in these trails covered longer periods of time, up to several seconds.

Since coming to Cordoba I have experimented further with trails as a means of testing the steadiness of the atmosphere with a view to using this method in the selection of a site for our large reflector. I have also studied somewhat the effect on meridian observations.

That these bodily displacements of the star's image result from irregular refraction in the atmosphere, probably by quite large masses, seems clear.

My experience at Mt. Hamilton and here at Cordoba leads me to think that there may sometimes be horizontal components to currents, especially in regions of peculiar surface contours, sufficient to cause a star's image to be displaced bodily for a considerable length of time (several minutes or perhaps even for hours). High winds may also have such an effect.

Short period relative displacements may be almost, if not entirely, eliminated by integrating the observations over a sufficient time, as by photography. Visual observations may be *improved* by the same means but to less extent generally than by photography.

It seems to me that we have reached a point in our most refined observations of position (particularly in M.C. work) where we are beginning to encounter the *variable* effects of refraction over comparatively long periods of time, and steps must be taken to avoid their injurious effects. It would seem impossible from the nature of the case to determine the amount of such effects and correct for them, and that the only practical way is to eliminate them as far as possible by seeking conditions which are free from such influences in which to make such observations.

I have been led to the following tentative conclusions, as a working hypothesis:—

1. The lighter the atmosphere (the higher the observing station) the less these effects will be, other things being equal.

2. The more nearly the observer is in the free air the less likely are irregularities of refraction to be met with.

These two conditions presuppose a freedom from violent winds which would cause the same class of difficulties as a dense atmosphere or currents.

3. That where a high degree of accuracy is sought, "instantaneous" images or settings should be avoided and an integrated result obtained extending over a considerable time.

4. That it is just as important to have a meridian circle, a vertical circle or zenith telescope which is intended for work of the highest possible accuracy, at a considerable altitude, as a great reflector or refractor — that fine atmospheric conditions should be as carefully sought for the one as for the other.

Further studies of these problems are in progress and I expect soon to have the results from special observations which are being made at different altitudes in the mountains.

Cordoba, February 18, 1913.

A MEANS OF MINIMIZING THE EFFECT OF SHORT PERIOD DISPLACEMENTS DUE TO VARIABLE REFRACTION FOR THE TRANSIT CIRCLE.

By BENJAMIN BOSS.

All meridian circle observers are familiar with the short period variability which seems to sway the star for a more or less long period in zenith distance, so that bisection becomes a difficult task. To free the observed zenith distances from errors arising from such a cause some observers have taken two or more settings during transit, and thus minimized the effect. The disadvantage of such procedure is the unlikelihood of catching the star consistently on opposite sides of the true position.

To obviate this trouble a device has been in operation at the Dudley Observatory for some years whereby the variable effects of refraction may be integrated. A fixed wire is used slightly inclined to the horizontal, the inclination being about 6″.8 in 1050″. A star entering the field can be placed to one side of the wire by means of a slow motion conveniently manipulated by the observer. In its course across the field the star will gradually come to bisection and then move to the other side of the wire. The place where bisection takes place is noted by the observer, and recorded. The inclination is determined by bisections taken at two vertical wires situated about 45^s each side of the middle wire on a number of nights. Such observations should only be taken when the atmospheric conditions are very steady.

The advantage of such a system of zenith distance bisection is that the bisection is not made instantaneously. If the star sways in one direction for the period of a few seconds, then sways in the opposite direction, a mental allowance can be made for it, or otherwise speaking, the observer can estimate the star's bisection very closely by noting the star's position relative to the inclined wire before and after bisection.

Should the period of displacement continue for a longer time than it takes the star to transit, the method would be ineffective. The only remedy in such a case is to try to eliminate the errors introduced by observing on a number of nights under different conditions.

CORRIGENDUM.

In the article of Professor E. E. Barnard in *A. J.*, 649-650 on Observations of the Seven Inner Satellites of *Saturn*, the following correction should be made at the top of page 5.

For Feb. 11^d 6^h 9^m 56^s read 6^h 19^m 56^s.

CONTENTS.

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OCCULTATIONS OF STARS BY THE MOON,

OBSERVATIONS MADE WITH THE 26-INCH AND THE 12-INCH EQUATORIALS OF THE U. S. NAVAL OBSERVATORY,
[Communicated by Captain J. L. JAYNE, U. S. Navy, Superintendent.]

Date	Object	Phen.	Wash. Sid. Time	Wash. Mean Time	See'g	Instrum't	Power	Obs.	Remarks
			h m s	h m s					
1910 Dec. 17	*c Geminorum*	DB	9 29 15.6	15 44 53.6	f	26-inch	388	H	
17	*c Geminorum*	RD	10 33 12.7	16 48 40.2	f	26-inch	388	H	Light clouds at emersion.
17	*c Geminorum*	DB	9 29 15.1	15 44 53.1	f	12-inch	235	E	Probably early by $0^s.2$.
17	*c Geminorum*	RD	10 33 12.6	16 48 40.1	f	12-inch	235	E	Thin clouds.
20	46 *Leonis*	DB	6 21 55.8	12 26 16.8	p	26-inch	388	H	Early $0^s.3$. Windy.
20	46 *Leonis*	RD	6 51 42.5	12 55 58.5	p	26-inch	388	H	
20	46 *Leonis*	DB	6 22 3.8	12 26 24.7	b	12-inch	115	E	
20	46 *Leonis*	RD	6 51 42.6	12 55 58.6	f	12-inch	115	E	
1911 Jan. 5	ψ^3 *Aquarii*	DD	1 4 41.9	6 7 0.2	f	26-inch	400	H	Eye and ear.
5	ψ^3 *Aquarii*	RB	2 2 58 –	7 5 7 –	f	26-inch	250	H	Late $0^s.5$. Light clouds. Eye and ear.
5	ψ^3 *Aquarii*	DD	1 4 41.4	6 6 59.7	g	12-inch	115	E	
5	ψ^3 *Aquarii*	RB	2 2 59.5	7 5 8.3	g	12-inch	115	E	$0^s.2$ late. Haze.
5	B.D. −10°6095	DD	1 10 27.7	6 12 45.1	g	12-inch	115	E	
9	*o Arietis*	DD	4 1 2.8	8 47 8.5	p	26-inch	388	B	Windy.
9	*o Arietis*	RB	5 9 59.5	9 55 53.9	p	26-inch	175	B	Windy.
9	*o Arietis*	DD	4 1 2.8	8 47 8.5	f	12-inch	115	E	
9	*o Arietis*	RB	5 10 1.6	9 55 56.0	p	12-inch	115	E	
14	4 *Cancri*	DB	3 10 52.6	7 37 27.0	p	26-inch	175	B	
14	4 *Cancri*	DB	3 10 52.8	7 37 27.2	b	12-inch	235	E	
15	90 *H'. Cancri*	DB	8 31 53.7	12 53 39.6	f	26-inch	175	H	Windy.
15	90 *H'. Cancri*	DB	8 31 54.6	12 53 40.5	f	12-inch	115	B	
16	*η Leonis*	DB	6 36 46.6	10 54 55.5	g	26-inch	175	B	Haze. Steady. Good disappearance.
16	*η Leonis*	DB	6 36 46.6	10 54 55.4	e	12-inch	335	E	
16	*η Leonis*	RD	7 45 42.3	12 3 39.8	e	12-inch	115	E	Possibly $0^s.1$ late.
Feb. 10	*c Geminorum*	DD	9 42 15.5	12 21 36.2	g	26-inch	175	B	
10	*c Geminorum*	RB	10 45 12.8	13 24 23.2	g	26-inch	175	B	
10	*c Geminorum*	DD	9 42 15.8	12 21 36.5	e	12-inch	500	E	
10	*c Geminorum*	RB	10 45 14.1	13 24 24.5	e	12-inch	500	E	Late 1^s.
13	46 *Leonis*	DB	5 42 33.0	8 10 45.2	p	26-inch	175	B	Haze.
13	46 *Leonis*	RD	6 48 13.8	9 16 15.2	g	26-inch	175	B	
13	46 *Leonis*	DB	5 42 26.4	8 10 38.6	p	12-inch	160	E	
13	46 *Leonis*	RD	6 48 13.9	9 16 15.3	g	12-inch	160	E	
Apr. 15	147 *B. Libræ*	DB	11 26 38.3	9 54 3.8	b	12-inch	115	E	Very uncertain.
15	147 *B. Libræ*	RD	12 30 47.8	10 58 2.7	p	26-inch	180	B	Haze.
May 15	10 *G. Sagittarii*	RD	19 51 22.5	16 19 28.0	g	12-inch	115	B	Chronograph record poor.
19	35 *Capricorni*	DB	18 20 22.1	14 32 58.9	f	26-inch	365	E	
19	35 *Capricorni*	RD	19 10 50.3	15 23 18.8	g	26-inch	365	E	
June 2	42 *Leonis*	DD	12 55 32.2	8 13 59.4	g	12-inch	235	B	
July 18	*o Piscium*	RD	21 44 55.9	14 1 4.4	–	26-inch	180	E	
Aug. 9	37 *Capricorni*	DB	23 12 1.6	14 1 25.9	f	26-inch	495	E	
9	37 *Capricorni*	RB	0 27 51.6	15 17 3.4	f	26-inch	495	E	
Sept. 6	161 *B. Capricorni*	DD	20 28 1.3	9 27 46.9	g	26-inch	495	E	
6	161 *B. Capricorni*	RB	21 53 32.2	10 53 3.8	g	26-inch	495	E	$0^s.2$ late.
1912 Jan. 27	45 *Arietis*	DD	5 45 6.3	9 21 5.4	f	26-inch	180	H	
27	45 *Arietis*	RB	6 54 27.3	10 30 15.1	f	26-inch	180	H	Light clouds. Late 2^s.

Date	Object	Phen.	Wash. Sid. Time	Wash. Mean Time	See'g	Instrum't	Power	Obs.	Remarks
1912			h m s	h m s					
Jan. 27	45 *Arietis*	DD	5 15 6.2	9 21 5.3	g	12-inch	160	B	
27	45 *Arietis*	RB	6 51 26.1	10 30 14.2	p	12-inch	160	B	Thin clouds. Perhaps a few seconds late.
31	134 *B. Geminorum*	RB	10 25 0.1	13 41 29.8	f	26-inch	180	H	Late about 1s.
31	134 *B. Geminorum*	DD	9 41 46.2	13 1 22.9	f	12-inch	160	W	
31	134 *B. Geminorum*	RB	10 24 59.7	13 44 29.4	f	12-inch	160	W	Probably 0s.5 to 1s.0 late.
Feb. 4	308 *B. Leonis*	DB	5 48 57.8	8 53 29.0	p	26-inch	180	H	Windy
4	308 *B. Leonis*	DB	5 48 53.8	8 53 25.0	p	12-inch	115	W	
28	4 *Cancri*	DD	13 30 55.1	14 59 48.8	p	12-inch	115	W	Hazy.
Mar. 27	ν' *Cancri*	DD	8 44 45.1	8 24 20.3	f	12-inch	115	W	Cloudy.
Apr. 4	169 *B. Libræ*	DB	13 1 1.6	12 8 27.5	p	26-inch	180	B	
4	169 *B. Libræ*	RD	14 16 55.9	13 24 9.4	f	26-inch	180	B	
4	169 *B. Libræ*	DB	13 1 2.2	12 8 28.1	p	12-inch	235	W	
4	169 *B. Libræ*	RD	14 16 57.2	13 24 10.7	f	12-inch	115	W	Probably 1s or 2s late.
4	177 *B. Libræ*	RD	14 44 15.4	13 51 24.4	f	26-inch	180	B	
4	177 *B. Libræ*	RD	14 44 15.6	13 51 24.6	p	12-inch	115	W	
4	42 *Libræ*	DB	15 44 36.6	14 51 35.7	f	26-inch	180	H	Early probably 1s on account of mountains.
4	42 *Libræ*	RD	15 57 42.5	15 4 39.5	f	26-inch	180	H	Earlier than predicted time by 4m or 5m.
4	42 *Libræ*	DB	15 44 41.9	14 51 41.0	p	12-inch	235	W	
23	λ *Cancri*	RB	13 48 26.3	11 41 2.2	g	26-inch	180	B	
23	λ *Cancri*	DD	13 5 16.6	10 57 59.6	f	12-inch	—	W	Taken without micrometer. Driving clock not running.
23	λ *Cancri*	RB	13 48 27.3	11 41 3.2	p	12-inch	—	W	Probably 0.s7 late. Taken without microm., clock not running.
May 8	37 *Capricorni*	DB	18 49 21.7	15 42 9.6	p	26-inch	180	B	
8	37 *Capricorni*	DB	18 49 24.9	15 42 12.8	p	12-inch	235	W	
20	ω *Cancri*	DD	13 13 6.3	9 19 38.4	f	12-inch	115	W	
20	ω *Cancri*	RB	13 28 58.2	9 35 27.7	f	12-inch	115	W	1s to 2s late.
20	4 *Cancri*	DD	13 21 37.6	9 28 8.3	f	12-inch	115	W	
20	4 *Cancri*	RB	14 14 6.7	10 20 28.8	p	12-inch	115	W	2s or 3s late.
23	*l Leonis*	RB	12 58 27.8	8 53 14.6	g	26-inch	180	H	
29	31 *B. Scorpii*	DD	17 38 37.4	13 9 2.8	–	26-inch	180	H	
29	31 *B. Scorpii*	DD	17 38 37.7	13 9 3.1	p	12-inch	115	W	
July 27	ω *Sagittarii*	DD	22 12 51.5	13 50 33.2	f	12-inch	115	W	
27	ω *Sagittarii*	RB	23 11 3.1	14 48 35.2	p	12-inch	115	W	
Aug. 3	ζ *Piscium*	DB	20 39 1.2	11 49 26.9	f	26-inch	180	H	
3	ζ *Piscium*	RD	21 28 48.3	12 39 5.8	f	26-inch	180	H	
3	ζ *Piscium*	DB	20 39 1.7	11 49 27.4	f	12-inch	115	W	
3	ζ *Piscium*	RD	21 28 48.5	12 39 6.0	f	12-inch	115	W	
3	*A.G.* Leipzig II, 433	DB	20 39 38.1	11 50 3.7	f	26-inch	180	H	
3	*A.G.* Leipzig II, 433	RD	21 29 49.2	12 40 6.5	f	26-inch	180	H	
3	*A.G.* Leipzig II, 433	DB	20 39 42.3	11 50 7.8	f	12-inch	115	W	
3	*A.G.* Leipzig II, 433	RD	21 29 49.4	12 40 6.7	f	12-inch	115	W	
Sept. 5	49 *Aurigæ*	DB	0 18 11.8	13 18 16.6	g	26-inch	180	B	
5	49 *Aurigæ*	RD	1 8 9.1	14 8 5.6	g	26-inch	180	B	
5	49 *Aurigæ*	DB	0 18 12.0	13 18 16.8	p	12-inch	160	W	
5	54 *Aurigæ*	DB	2 18 53.8	15 18 38.8	g	12-inch	115	W	
5	54 *Aurigæ*	RD	2 42 1.9	15 41 13.1	g	12-inch	115	W	
5	*B.D.* +28°1201	DB	2 20 18.0	15 20 2.8	f	26-inch	180	B	
5	*B.D.* +28°1201	RD	3 22 34.0	16 22 8.6	f	26-inch	180	B	
29	ζ *Arietis*	DB	23 39 39.7	11 5 29.0	g	12-inch	115	W	Haze and clouds.
29	ζ *Arietis*	RD	0 39 58.0	12 5 37.1	f	12-inch	115	W	Slightly hazy.
Oct. 1	354 *B. Tauri*	DB	5 6 13.7	16 23 47.6	f	26-inch	180	B	
1	354 *B. Tauri*	RD	6 27 22.6	17 44 13.3	g	26-inch	180	B	
1	354 *B. Tauri*	DB	5 6 40.4	16 23 14.3	f	12-inch	115	W	Somewhat uncertain.
1	354 *B. Tauri*	RD	6 27 22.7	17 44 13.4	g	12-inch	115	W	Nearly daylight.
4	28 *Cancri*	DB	4 16 39.1	15 22 3.4	f	26-inch	180	B	
4	28 *Cancri*	RD	5 13 20.9	16 18 35.9	f	26-inch	180	B	
4	28 *Cancri*	DB	4 16 39.9	15 22 4.2	p	12-inch	235	W	
4	28 *Cancri*	RD	5 13 20.9	16 18 36.0	f	12-inch	115	W	Dark limb visible.
4	ν' *Cancri*	DB	5 55 40.4	17 0 18.5	g	26-inch	180	B	
4	ν' *Cancri*	RD	6 25 58.4	17 31 1.5	g	26-inch	180	B	
4	ν' *Cancri*	DB	5 55 42.1	17 0 50.5	p	12-inch	235	W	
4	ν' *Cancri*	RD	6 25 58.5	17 31 1.6	f	12-inch	115	W	Nearly daylight. Dark limb visible.
25	19 *Arietis*	DB	5 17 23.4	15 0 3.7	g	12-inch	115	W	

Date		Object	Phen.	Wash. Sid. Time	Wash. Mean Time	See'g	Instrum't	Power	Obs.	Remarks
				h m s	h m s					
1912 Oct.	25	19 *Arietis*	RD	6 27 49.4	16 10 18.2	g	12-inch	160	W	0ˢ.2 or 0ˢ.3 late.
	27	104 *B. Tauri*	DB	22 16 45.8	7 52 43.1	f	26-inch	180	B	
	27	104 *B. Tauri*	RD	22 51 6.7	8 26 58.4	f	26-inch	180	B	
	27	104 *B. Tauri*	RD	22 51 7.6	8 26 59.4	f	12-inch	115	W	Possibly a little late.
	29	406 *B. Tauri*	RD	23 36 34.1	9 4 26.6	p	26-inch	180	B	Thick haze. Reappearance seemed indefinite.
	29	406 *B. Tauri*	RD	23 36 32.6	9 4 25.2	f	12-inch	115	W	Reappearance possibly delayed by thick haze.
Nov.	20	171 *B. Piscium*	DD	2 37 57.2	10 38 49.9	g	26-inch	180	B	Perhaps a trifle late.
	20	171 *B. Piscium*	RB	3 55 13.7	11 55 53.8	g	26-inch	180	B	
	20	171 *B. Piscium*	DD	2 37 58.8	10 38 51.5	f	12-inch	115	W	A little late.
	20	171 *B. Piscium*	RB	3 55 20.0	11 56 0.4	f	12-inch	335	W	
	25	107 *B. Aurigæ*	DB	2 36 59.8	10 18 13.1	f	26-inch	388	B	
	25	107 *B. Aurigæ*	RD	3 15 46.3	10 56 53.3	p	26-inch	388	B	Haze.
	25	107 *B. Aurigæ*	DB	2 36 54.6	10 18 8.0	p	12-inch	335	W	
	25	107 *B. Aurigæ*	RD	3 15 43.3	10 56 50.3	p	12-inch	115	W	Hazy. A little late. Star very faint.
Dec.	19	19 *Arietis*	DD	3 3 18.5	9 10 5.6	f	26-inch	388	B	
	19	19 *Arietis*	RB	4 6 45.7	10 13 22.4	p	26-inch	388	B	Thin clouds over Moon at reappearance.
	24	47 *Geminorum*	RD	1 10 35.8	6 58 1.9	p	26-inch	180	B	Perhaps a trifle late.
	24	47 *Geminorum*	RD	1 10 37.3	6 58 3.3	p	12-inch	115	W	A little late.
	25	λ *Cancri*	DB	1 58 51.9	7 42 14.1	p	26-inch	180	B	Not a good disappearance.
	25	λ *Cancri*	RD	2 51 25.6	8 34 39.1	f	26-inch	180	B	
	25	λ *Cancri*	DB	1 58 51.5	7 42 13.7	f	12-inch	115	W	Rather uncertain.
	25	λ *Cancri*	RD	2 51 25.6	8 34 39.2	f	12-inch	115	W	
	28	σ *Leonis*	RD	8 0 37.2	13 31 12.4	g	26-inch	180	B	
	28	σ *Leonis*	DB	7 5 57.8	12 36 42.0	p	12-inch	160	W	
	28	σ *Leonis*	RD	8 0 37.3	13 31 12.5	f	12-inch	115	W	Dark limb visible.
	30	*g Virginis*	DB	12 53 31.6	18 15 26.9	f	26-inch	388	B	
	30	*g Virginis*	RD	14 12 46.1	19 34 28.5	p	26-inch	388	B	Daylight and thin clouds.
1913 Jan.	19	406 *B. Tauri*	DD	10 32 4.0	14 35 44.3	f	26-inch	388	B	
	19	406 *B. Tauri*	DD	10 32 4.0	14 35 44.3	p	12-inch	—	W	
	21	4 *Cancri*	DD	9 13 26.0	13 9 27.4	g	26-inch	388	B	
	21	4 *Cancri*	RB	10 18 18.3	14 14 9.1	g	26-inch	388	B	Probably a trifle late.
	21	4 *Cancri*	DD	9 13 26.2	13 9 27.6	g	12-inch	115	W	
	21	4 *Cancri*	RB	10 18 19.0	14 14 9.8	g	12-inch	115	W	Late.
	25	*β Virginis*	DB	6 54 44.7	10 35 25.1	f	12-inch	160	W	
Feb.	15	47 *B. Aurigæ*	DD	5 58 46.7	8 17 2.1	g	12-inch	115	W	
	15	47 *B. Aurigæ*	RB	6 32 17.1	8 50 27.1	g	12-inch	335	W	Rather late.
	15	354 *B. Tauri*	DD	11 16 20.2	13 33 43.7	f	26-inch	180	B	Trifle hazy.
	15	354 *B. Tauri*	RB	11 42 2.9	13 59 22.2	p	26-inch	180	B	Observation late.
	15	354 *B. Tauri*	DD	11 16 20.2	13 33 43.7	g	12-inch	115	W	
	15	354 *B. Tauri*	RB	11 42 7.0	13 59 26.2	p	12-inch	115	W	
	22	162 *B. Virginis*	DB	15 24 20.1	17 13 31.6	p	12-inch	235	W	
	22	162 *B. Virginis*	RD	15 36 8.0	17 25 17.6	p	12-inch	115	W	A little late.
Feb.	25	47 *G. Libræ*	DB	13 16 20.8	14 54 5.5	p	26-inch	388	B	Haze. Disappearance gradual.
	25	47 *G. Libræ*	RD	14 29 18.9	16 6 51.7	p	26-inch	388	B	Haze.
	25	47 *G. Libræ*	DB	13 16 20.2	14 54 4.9	f	12-inch	235	W	Hazy.
	25	47 *G. Libræ*	RD	14 29 19.2	16 6 51.9	f	12-inch	115	W	Hazy.
Mar.	12	47 *Arietis*	RB	7 13 53.4	7 53 38.9	f	26-inch	388	B	Haze. Steady.
	12	47 *Arietis*	DD	6 6 17.6	6 46 14.2	g	12-inch	115	W	Good disappearance. Haze.
	12	47 *Arietis*	RB	7 14 2.4	7 53 47.9		12-inch	235	W	Late on account of thick haze. Steady.
	17	ω *Cancri*	RB	10 3 30.5	10 23 8.6	p	12-inch	115	W	Late.
	17	4 *Cancri*	DD	9 41 3.9	10 0 45.7	f	12-inch	115	W	
	17	4 *Cancri*	RB	10 52 39.5	11 12 9.6	p	12-inch	115	W	
	22	319 *B. Virginis*	DB	11 4 8.8	11 3 57.4	f	26-inch	388	B	
	22	319 *B. Virginis*	RD	11 41 39.7	11 41 22.2	f	26-inch	388	B	Observation late.
	22	319 *B. Virginis*	DB	11 4 4.1	11 3 52.7	f	12-inch	235	W	
Apr.	17	89 *Leonis*	DD	15 13 47.8	13 30 41.9	f	12-inch	115	W	
	17	89 *Leonis*	RB	16 10 27.0	14 27 11.8	f	12-inch	115	W	
	21	47 *G. Libræ*	DB	12 32 24.6	10 34 1.6	f	26-inch	388	B	
	21	47 *G. Libræ*	RD	13 43 13.8	11 44 39.1	f	26-inch	388	B	
	21	47 *G. Libræ*	DB	12 32 22.4	10 33 59.4	p	12-inch	160	W	
	21	47 *G. Libræ*	RD	13 43 13.9	11 44 39.2	f	12-inch	115	W	
	22	48 *B. Scorpii*	RD	12 48 25.8	10 46 4.3	f	26-inch	180	B	

Date	Object	Phen.	Wash. Sid. Time	Wash. Mean Time	See'g	Instrum't	Power	Obs.	Remarks
1913			h m s	h m s					
Apr. 22	65 *B. Scorpii*	DB	14 17 20.4	12 14 44.3	p	26-inch	388	B	Diasppearance gradual.
22	65 *B. Scorpii*	RD	15 33 31.6	13 30 43.0	f	26-inch	388	B	
22	65 *B. Scorpii*	DB	14 17 17.4	12 14 41.3	p	12-inch	160	W	Hazy.
22	65 *B. Scorpii*	RD	15 33 31.7	13 30 43.1	f	12-inch	115	W	
29	182 *B. Aquarii*	DB	18 48 55.3	16 18 3.3	p	26-inch	180	H	Disappearance gradual.
29	182 *B. Aquarii*	DB	18 48 49.9	16 17 57.9	p	12-inch	115	W	

In the third column DD signifies star disappeared at dark limb of Moon; DB, disappeared at bright limb; RD, reappeared at dark limb; RB, reappeared at bright limb. In sixth column, under Seeing, e = excellent; g = good; f = fair; p = poor; b = bad.

All observations were recorded on chronograph except as noted under Remarks.

Observers: H = A. Hall; E = J. B. Eppes; B = H. E. Burton and W = C. B. Watts.

OBSERVATIONS OF VARIABLE STARS,—No. 14,

By WM. E. SPERRA.

103. *T Andromedæ.*

Twelve observations of this star, extending from 1908 January 6 to 1908 March 9, give as the date of maximum 1908 February 10, at $7^m.5$. At first observation the star was of $9^m.6$, and at last observation had declined to $8^m.3$. All but two observations on the decline, lie close to the curve as drawn, and they deviate three light steps, one each above and below.

A maximum for 1908 November 26, at $8^m.6$, is indicated by six observations, 1908 November 13, at $9^m.0$, to 1908 December 27, when it had declined to $9^m.5$.

A third maximum is indicated for 1912 September 24, at $8^m.4$, from seven observations between 1912 August 14, when at $11^m.0$, and 1912 October 27, when the star's light had declined to $9^m.3$.

The dates of all three maxima are well determined, as the observations are very accordant with the exception noted in first.

294. *W Cassiopeiæ.*

Twenty observations 1908 June 23 to 1908 November 28, of this star yield as the date for maximum, 1908 October 4, at $8^m.9$. The star was at $9^m.3$ at beginning of series, and with the exception of considerable fluctuation, had attained only $9^m.2$ by September 1, when the rise really commenced. The rise and decline were of nearly equal steepness, and observations at this part of the curve were quite accordant. $9^m.3$ was reached at the end of the series.

1623. *T Camelopardalis.*

This star was observed on eighteen dates between 1907 December 7 and 1908 April 25, and indicate as the time of maximum 1908 March 7, at $7^m.9$. From $9^m.2$ at first observation, the rise was steady, and after passing a sharp crest, the decline was much steeper than the rise, and $8^m.8$ was reached at the last observation.

4521. *R Virginis.*

Sixteen observations of this star, between 1908 April 4 and 1908 July 11, indicate as the time of maximum 1908 June 20, at $6^m.9$. The star was at $8^m.0$ at beginning of the series and at the end had reached $8^m.3$. For about ten days the light was stationary at maximum, and all observations lie in or very close to the smoothed curve.

5768. *RR Herculis.*

This star was observed twenty-one times between 1908 April 4, when at $8^m.8$, and 1908 October 13, when $8^m.6$ was reached on the decline, and the indicated date for maximum is 1908 July 24, at $7^m.9$. The rising phase was marked by a stationary light May 10 to June 17, amounting to a flat secondary crest at $8^m.3$ for May 21. A break in the series of observations between June 17 and July 19 causes several days' uncertainty in the time of passing maximum.

6207. *Z Ophiuchi.*

This star was observed twenty-seven times between 1908 June 27, when at $10^m.5$, and 1908 November 15, when $8^m.6$ was reached on the decline. The date of passing maximum is indicated quite definitely for 1907 August 29, at $7^m.6$.

7085. *RT Cygni.*

What follows is in continuation of what was published in *A.J.*, 606, and with the exception of several minima and one maxima, when the star was unfavorably located for observation, is continuous, and in all there are seventy-two observations, 1909 April 24 to 1911 October 11.

A maximum for 1909 July 14, at $6^m.9$, is based on fourteen observations, June 15—August 30. The rise was rapid, and the decline slow for five weeks after passing maximum, or until $7^m.9$ was reached, and then, in less than three weeks it dropped to $9^m.9$, and then slowed off for the minimum.

The minimum was passed 1909 October 26, at $11^m.5$, based on nine observations, September 5—December 6. The stay at minimum was short, and the rise was well under way the first week in November, and continued rapid until maximum was reached.

A maximum is indicated for 1910 January 16, at $6^m.9$, from five observations, 1909 December 19 to 1910 March 5, when at $9^m.5$.

The next minimum was not sufficiently observed, but must have occurred about the last week in April. The

rise to maximum was very slow requiring about three months.

The next maximum is indicated for 1910 July 30, at $6^m.8$, based on fifteen observations June 24—September 4. Besides the very slow rise, already referred to, the star remained for nearly a month at or near maximum, followed by a rapid decline, which started the last week in August, and in November observations were discontinued as it passed below my limit, $11^m.5$.

The next maximum and minimum were missed, observations not commencing until 1911 June 1, when the star was marked as very faint.

A maximum for 1911 August 7, at $7^m.0$, is based on sixteen observations, July 3—September 25. The rise was slow, followed by a very sharp crest, and a somewhat slow irregular decline.

7106. *S Vulpeculæ.*

Twenty-nine observations from 1908 March 10 to 1908 September 29, yield results for this star as follows:

Maximum	1908	May 15	8.7	Fairly good.
Minimum		July 3	9.8	Good
Maximum		Aug. 1	8.8	Very good.
Minimum		Sept. 10	9.8	Fairly good.
Maximum		Oct. 3?	8.7	

Cleveland, Ohio, 1913, *July* 20.

OBSERVATIONS OF BORRELLY'S COMET 1911 *e*,

MADE WITH THE 12-INCH REFRACTOR OF THE ARGENTINE NATIONAL OBSERVATORY, CORDOBA,

By E. CHAUDET.

Greenwich M.T.	*	Comp.	$\Delta\alpha$	$\Delta\delta$	App. α	App. δ	log $p\Delta$ α	log $p\Delta$ δ	Red. to App. Pl. α	Red. to App. Pl. δ
1911 d h m s			m s	′ ″	h m s	° ′ ″			s	″
Oct. 23 16 12 49	1	13 12	+0 16.16	− 3 23.8	3 53 31.47	−33 31 26.9	9.450*n*	9.133*n*	+2.98	+24.1
24 15 5 26	1	11 15	−0 22.89	+ 4 24.8	3 52 52.44	−33 23 38.6	9.618*n*	9.912*n*	+3.00	+23.8
26 16 9 16	2	6 6	−0 25.84	− 0 35.1	3 51 19.21	−33 4 14.8	9.408*n*	9.058*n*	+3.05	+23.4
Nov. 1 15 21 46	4	12 10	−0 6.26	+ 4 3.6	3 45 32.31	−31 43 33.9	9.467*n*	9.654*n*	+3.18	+21.9
2 19 28 26	6	8 8	−0 3.16	− 0 37.8	3 44 11.25	−31 23 8.6	9.508	9.794*n*	+3.19	+21.7
13 14 27 19	8	8 10	+0 9.83	+ 3 26.3	3 29 25.40	−26 48 48.0	9.416*n*	0.028*n*	+3.35	+19.0
17 14 11 25	10	10 10	+0 1.74	+ 6 28.1	3 23 10.72	−24 23 28.9	9.384*n*	0.136*n*	+3.39	+18.2
18 18 30 53	11	2 4	+0 0.92	+ 4 1.9	3 21 16.36	−23 35 31.7	9.549	0.276*n*	+3.39	+18.0
19 14 5 51	13	10 10	+0 13.70	+10 15.4	3 19 59.26	−23 1 10.8	9.358*n*	0.124*n*	+3.40	+17.8
22 14 20 57	14	8 7	+0 23.41	+ 4 13.5	3 15 10.31	−20 44 58.4	9.190*n*	0.235*n*	+3.42	+17.4
26 17 39 53	16	14 10	−0 17.62	+ 0 30.6	3 8 41.16	−17 14 47.8	9.515	0.417*n*	+3.43	+17.0
27 14 29 31	18	6 10	+0 10.47	− 6 12.6	3 7 22.63	−16 27 33.7	8.850*n*	0.355*n*	+3.43	+16.9
27 18 12 42	18	10 6	−0 4.29	+ 2 21.1	3 7 7.87	−16 19 0.0	9.590	0.476*n*	+3.43	+16.9
29 15 37 14	19	4 4	−3 37.70	+ 2 2.8	3 4 20.44	−14 32 8.3	9.029	0.410*n*	+3.44	+16.8
30 19 45 11	20	6 6	+1 18.89	+ 0 50.4	3 2 38.79	−13 23 30.1	9.698	0.602*n*	+3.44	+16.8
Dec. 8 16 1 31	21	.. 6	....	− 2 43.6	....	− 5 10 36.9	...	0.597*n*	+3.45	+17.3
8 16 31 37	21	8	−0 18.65	....	2 52 44.73	....	9.484	....	+3.45	+17.3
9 16 16 34	23	3 ..	−0 31.49	....	2 51 41.62	....	9.458	....	+3.45	+17.4
9 15 56 6	23	. 6	....	− 1 4.7	....	− 4 5 0.4	...	0.611*n*	+3.45	+17.4
11 16 13 56	24	10 10	+0 6.74	− 1 18.1	2 49 43.81	− 1 51 30.1	9.476	0.640*n*	+3.45	+17.7
14 15 27 4	25	5 4	−0 25.69	+ 1 8.9	2 47 12.75	+ 1 26 30.8	9.378	0.675*n*	+3.46	+18.3
15 14 45 56	27	.. 4	....	− 1 54.4	....	+ 2 31 14.1	...	0.687*n*	+3.46	+18.4
27 15 28 44	28	5 12	−0 42.08	+ 8 15.4	2 42 25.71	+15 16 49.8	9.548	0.767*n*	+3.50	+21.3

Mean Places for 1911.0 of the Comparison Stars.

*	α	δ	Authority	*	α	δ	Authority
	h m s	° ′ ″			h m s	° ′ ″	
1	3 53 12.33	−33 28 27.2	Cordoba meridian observ. C.D. −33° 1460	15	3 17 42.05	−20 38 34.8	Arg. Gen. Cat. 3644
2	3 51 42.00	−33 4 3.1	C.D. −33° 1443 micr. comp. with Star 3	16	3 8 55.35	−17 15 35.4	S.D. −17° 619 micr. comp. Star 17
3	3 52 10.33	−32 55 29.1	Cor. Zone Cat. 3^h 1570	17	3 8 29.62	−17 6 41.7	A.G. Cat. Wash. 848
4	3 45 35.39	−31 47 59.4	C.D. −31° 1555 micr. comp. Star 5	18	3 7 8.73	−16 21 38.0	A.G. Cat. Wash. 840
5	3 40 19.99	−31 43 14.2	Arg Gen. Cat. 4136	19	3 7 54.70	−14 34 27.9	A.G. Cat. Wash. 844
6	3 44 11.22	−31 22 52.5	C.D. −31° 1548 micr. comp. Star 7	20	3 1 16.46	−13 24 37.3	A.G. Cat. Cambr 700
7	3 39 35.47	−31 18 4.1	Arg. Gen. Cat. 4118	21	2 52 59.93	− 5 8 10.6	S.D. −5° 548 micr. comp. Star 22
8	3 29 13.12	−26 52 33.3	C.D. −26° 1337 micr. comp. Star 9	22	2 53 30.50	− 5 4 52.2	A.G. Cat. Str. 711
9	3 28 52.60	−27 3 14.7	C.D. −27° 1269 Cordoba merid. observ.	23	2 52 9.66	− 4 4 13.1	Pr. Gen. Cat. Boss 1900.0 666
10	3 23 5.59	−24 30 15.2	C.D. −24 1657 Cord. New Catal. 1900.0 717	24	2 49 33.62	− 1 50 29.7	A.G. Cat. Str. 696
11	3 21 12.05	−23 39 51.6	C.D. −23° 1377 micr. comp. Star 12	25	2 47 34.98	+ 1 25 3.6	B.D. +1° 505 micr. comp. Star 26
12	3 25 41.80	−23 46 57.6	Arg. Gen. Cat. 3812	26	2 47 39.16	+ 1 33 48.6	A.G. Cat. Albany 804
13	3 19 42.16	−23 11 44.0	Cord. New Cat. 1900.0 705	27	2 47 15.47	+ 2 32 50.1	A.G. Cat. Albany 800
14	3 14 43.48	−20 49 29.3	S.D. −21° 601 micr. comp. Star 15	28	2 43 4.29	+15 8 13.1	A.G. Cat. Leipz. I 818

NOTES.

All the observations are direct measures with the micrometer, except those of November 29th and 30th.

Oct. 23. Sky bad, identification of faint stars almost impossible.

Oct. 24. Sky bad, comet disappears sometimes, star images blurred.

Oct. 26. Very hazy, seeing bad, images blurred.

Nov. 1. Sky good. Moonlight interferes; comet *not* visible in the finder, illumination of the wires deficient. Probable magnitude of the nucleus, $9\frac{3}{4}$–10.

Nov. 2. Sky very bad, observation frequently interrupted by clouds. Moonlight and strong lightning in the west and north interferes considerably.

Nov. 27. Hazy for the first observation and very hazy for the second observation.

Nov. 30. Very hazy, images blurred.

Dec. 8. Sky bad, too hazy and damp, moonlight. Observation interrupted by clouds.

Dec. 14. Sky very bad, very hazy; observation interrupted by clouds.

Dec. 15. Sky very bad, observation interrupted by passing clouds. Measures in α impossible, it clouded over. Comet often invisible.

Dec. 27. Observations in α very difficult, very windy, images blurred.

OBSERVATIONS OF COMET 1911 *g* (*BELJAWSKY*),

MADE WITH THE 12-INCH REFRACTOR OF THE ARGENTINE NATIONAL OBSERVATORY,

By C. D. PERRINE.

Greenwich M.T.	✱	Comp.	Δα	Δδ	App. α	App. δ	log p Δ α	log p Δ δ	Red. to App. Pl. α	Red. to App. Pl. δ
1912 d h m s			m s	′ ″	h m s	° ′ ″			s	″
Jan. 28 20 15 40	1	10 8	+0 13.73	− 2 50.4	18 47 5.34	−44 19 40.5	9.845n	0.569n	−1.78	− 6.1
Feb. 14 20 31 56	3	10 8	−0 17.32	+ 1 41.4	19 9 9.33	−47 35 19.0	9.864n	0.315n	−1.35	− 4.3
15 20 18 18	5	10 8	−0 5.92	+ 4 41.2	19 10 19.19	−47 47 7.0	9.869n	0.365n	−1.32	− 4.1
16 20 23 36	7	10 8	+0 38.55	+ 2 13.7	19 11 29.86	−47 59 5.8	9.868n	0.323n	−1.28	− 4.0

Mean Places of the Comparison Stars for 1912.0.

✱	α	δ	Authority	✱	α	δ	Authority
	h m s	° ′ ″			h m s	° ′ ″	
1	18 46 53.39	−44 16 44.0	Mic. comp. with star 2	5	19 10 26.43	−47 51 44.1	10th magn. star micr. comp. with Star 6
2	18 45 57.57	−44 26 24.1	Arg. Gen. Cat. 25763	6	19 8 39.49	−48 1 28.8	Gen. Arg. Cat. 26316
3	19 9 28.00	−47 36 56.1	Mic. comp. with star 4	7	19 10 52.59	−48 1 15.5	$9\frac{1}{2}$ mag. star micr. comp. with Star 6
4	19 7 23.00	−47 30 21.7	Arg. Gen. Cat. 26289				

NOTES.

Jan. 28. ☄ 11th magnitude, diameter about 3′. Diffuse. Slight condensation, extrusion away from the Sun.

Feb. 14. ☄ about 3′ in diameter, very diffuse, very little condensation. Very difficult to measure. Seeing 3.

All the observations are direct measures.

Feb. 15. ☄ very difficult object, about 12th magnitude. Seeing 3, sky good.

Feb. 16. ☄ very faint and difficult, 3′ or 4′ in diameter and scarcely a trace of any condensation.

Observations reduced by Señor Chaudet.

OBSERVATIONS OF COMET 1911 *f* (*QUENISSET*),

MADE WITH THE 12-INCH REFRACTOR OF THE ARGENTINE NATIONAL OBSERVATORY, CORDOBA.

Greenwich M.T.	✱	Comp.	Δα	Δδ	App. α	App. δ	log p Δ α	log p Δ δ	Red. to App. Pl. α	Red. to App. Pl. δ
1911 d h m s			m s	′ ″	h m s	° ′ ″			s	″
Dec. 30 19 36 59	1	4 4	−0 19.21	+13 47.9	15 49 44.60	−28 42 7.2	9.757n	0.594n	+1.80	−10.2*
1912 Jan. 5 19 18 14	2	2 2	+0 15.05	− 0 55.8	15 49 30.90	−32 41 44.8	9.774n	0.562n	−1.77	+ 1.1*
6 19 58 58	3	5 7	−0 9.64	+ 3 33.1	15 49 24.20	−33 21 56.9	9.766n	0.452n	−1.76	+ 1.1*
10 19 31 58	4	7 3	−0 27.63	− 2 23.7	15 48 47.37	−36 4 29.1	9.784n	0.453n	−1.68	+ 1.6*
16 19 56 36	5	3 5	+0 36.83	+ 3 23.3	15 47 0.69	−40 16 48.2	9.780n	0.186n	−1.54	+ 2.7*
17 20 20 37	6	1 2	+0 13.20	+ 3 17.4	15 46 35.32	−41 0 22.6	9.753n	9.937n	−1.52	+ 3.0*
[illegible] 16 19 10 18	7	10 4	−0 51.01	+ 6 43.1	14 55 22.77	−63 26 30.0	9.707n	9.596	−0.21	+ 7.7†

* E. Chaudet † C. D. Perrine

Mean Places of Comparison Stars for 1911.0 and 1912.0.

*	α	δ	Authority	*	α	δ	Authority
	h m s	° ′ ″			h m s	° ′ ″	
1	15 50 2.01	−28 55 44.9	Arg. Gen. Cat. 21562	5	15 46 25.40	−40 20 14.2	Cor. Zone Cat. 15h 3095
2	15 49 17.62	−32 40 50.1	Cor. merid. obs. 1912.0	6	15 46 23.64	−41 3 43.0	Cor. Zone Cat. 15h 3092
3	15 49 35.60	−33 25 31.1	Cor. merid. obs. 1912.0	7	14 56 14.02	−63 33 20.8	Cor. meridian circle obs. 1912.0
4	15 49 16.68	−36 2 7.0	Cor. merid. obs. 1912.0				

NOTES.

1911 Dec. 30. Magnitude of ☄ about 9th, weak nuclear condensation. Sky good.

1912 Jan. 5. Approximate measures only. Comet was seen only a few minutes. Comet about 9th magnitude. Very hazy. Moonlight.

Jan. 6. Sky bad, hazy. Comet about 9th magnitude.

All the observations are direct measures.

Jan. 10. Measures very difficult and uncertain. Moonlight interferes. Sky bad.

Jan. 16. Some measures uncertain. Comet very faint, probably 9¾ magnitude. Sky bad.

Jan. 17. Approximate measures only, atmosphere bad.

Feb. 16. ☄ about 11th magnitude, very diffuse, no nuclear condensation.

All of the reductions by Señor Chaudet.

OBSERVATIONS OF COMET 1911*c* (*BROOKS*),

MADE WITH THE 12-INCH REFRACTOR OF THE ARGENTINE NATIONAL OBSERVATORY, CORDOBA.

By E. CHAUDET.

Greenwich M.T.	*	Comp.	Δα	Δδ	App. α	App. δ	log pΔ α	log pΔ δ	Red. to App. Pl. α	Red. to App. Pl. δ
1911 d h m s			m s	′ ″	h m s	° ′ ″			s	″
Nov. 16 20 4 3	1	5 3	−0 14.98	+ 2 18.9	13 1 27.85	−18 26 9.9	9.720*n*	0.607*n*	+0.95	− 6.7
16 20 8 55	1	2 5	−0 14.65	+ 2 8.5	13 1 28.18	−18 26 20.3	9.719*n*	0.604*n*	+0.95	− 6.7
17 20 7 12	2	6 8	+0 33.95	+ 2 43.2	13 3 44.80	−19 19 43.1	9.721*n*	0.598*n*	+0.96	− 6.6
17 20 20 36	2	11 4	+0 35.12	+ 2 13.4	13 3 45.97	−19 20 12.9	9.717*n*	0.582*n*	+0.96	− 6.6
29 20 16 26	3	.. 8	....	− 4 18.4	....	−28 18 41.5	...	0.489*n*	+1.12	− 6.9
Dec. 27 17 46 23	4	.. 12	....	+ 2 8.2	....	−41 44 1.6	...	0.627*n*	.....	− 7.2
27 18 16 31	4	10 ..	+0 16.73	....	14 30 9.60	....	9.827*n*	...	+1.85	.....
27 19 1 0	4	.. 30	....	+ 0 57.4	....	−41 45 12.4	....	0.416*n*	.....	− 7.2
27 20 1 1	4	20 ..	+0 24.67	....	14 30 17.54	....	9.788*n*	....	+1.85	.....
28 18 50 15	5	7 6	+2 31.33	− 2 16.3	14 31 59.21	−42 7 6.1	9.826*n*	0.443*n*	+1.86	− 7.0
30 18 38 48	6	8 8	+0 14.40	+ 3 5.2	14 35 28.28	−42 50 7.8	9.833*n*	0.460*n*	+1.93	− 6.9
1912 31 19 32 47	7	. 3	....	− 9 29.2	....	−43 12 8.5	....	0.211*n*	+1.97	− 6.9
Jan. 1 19 39 10	8	6 7	−0 3.31	+ 5 11.9	14 38 55.84	−43 33 4.7	9.809*n*	0.150*n*	−1.87	+ 8.5
5 19 8 48	9	4 6	+0 38.56	− 7 12.0	14 45 22.53	−44 53 30.2	9.832*n*	0.244*n*	−1.77	+ 8.1
6 18 34 39	10	3 7	+0 2.42	+ 0 57.3	14 46 52.83	−45 12 37.8	9.848*n*	0.389*n*	−1.76	+ 7.9
11 18 43 51	11	5 8	+0 25.57	+ 1 52.4	14 54 12.63	−46 47 23.2	9.852*n*	0.263*n*	−1.64	+ 7.6
13 18 15 28	12	7 12	+0 24.49	+ 8 15.5	14 56 54.10	−47 23 33.6	9.866*n*	0.369*n*	−1.59	+ 7.6
13 19 7 57	13	4	....	+11 18.1	....	−47 24 16.7	...	0.045*n*	−1.59	+ 7.6
13 19 21 42	13	3 .	+0 4.25	....	14 56 58.22	....	9.829*n*	...	−1.59	+ 7.6
16 18 7 6	14	4 8	+0 6.98	+ 7 42.4	15 0 44.05	−48 16 53.7	9.873*n*	0.359*n*	−1.50	+ 7.4
17 19 56 29	16	3 7	+0 46.02	+ 9 8.3	15 2 2.68	−48 35 41.1	9.789*n*	9.444	−1.47	+ 7.3
20 19 45 10	17	3 4	−0 8.41	− 8 45.9	15 5 27.86	−49 26 54.3	9.800*n*	9.497	−1.38	+ 6.9
21 19 31 12	18	16 12	−0 17.82	− 6 23.4	15 6 31.73	−49 43 34.4	9.817*n*	9.053	−1.35	+ 6.9
24 18 56 37	19	6 8	−0 22.99	+ 1 47.4	15 9 30.43	−50 32 45.5	9.851*n*	9.570*n*	−1.28	+ 6.8
28 18 35 0	21	14 10	+0 34.16	− 8 10.1	15 12 58.75	−51 36 44.6	9.870*n*	9.687*n*	−1.11	+ 6.7
28 19 18 16	22	. 5	....	−11 2.8	....	−51 37 10.2	....	9.685	−1.10	+ 6.7
29 20 0 34	23	3 7	−0 10.54	− 5 18.8	15 13 47.47	−51 53 24.7	9.754*n*	0.151	−1.07	+ 6.6
Feb. 13 19 5 20	25	. 3	....	− 3 0.2	....	−55 33 47.5	...	0.258	−0.44	+ 5.7
13 19 29 49	25	2 .	−0 25.44	....	15 19 58.74	....	9.744*n*	...	−0.44	+ 5.7
14 19 6 50	26	6 7	+0 37.61	+ 6 58.2	15 19 59.64	−55 47 24.3	9.788*n*	0.287	−0.38	+ 5.7
15 18 29 12	26	12 10	+0 35.96	− 6 16.0	15 19 58.04	−56 0 38.7	9.849*n*	0.109	−0.33	+ 5.5
16 17 51 32	27	10 8	−0 13.96	+ 0 36.6	15 19 52.66	−56 13 44.4	9.894*n*	9.739	−0.28	+ 5.4
20 19 13 30	28	5 5	+0 3.53	− 7 44.8	15 18 58.22	−57 6 33.2	9.732*n*	0.423	−0.10	+ 5.1
21 20 36 11	29	4 8	−0 48.14	− 4 49.4	15 18 34.35	−57 20 2.6	9.348*n*	0.566	−0.04	+ 5.0
26 20 46 29	30	. 2	....	− 3 37.7	....	−58 20 21.6	...	0.599	+0.26	+ 4.7
27 18 14 44	31	3 6	−0 3.20	+ 1 54.4	15 15 9.91	−58 30 25.3	9.812*n*	0.368	+0.31	+ 4.5

All the observations are direct measures except that of December 28th.

Mean Places for 1911.0 and 1912.0 of the Comparison Stars.

*	α	δ	Authority	*	α	δ	Authority
	h m s	° ′ ″			h m s	° ′ ″	
1	13 1 11.88	−18 28 22.1	A.G.Cat. Wash. 1990	17	15 5 37.65	−49 18 15.3	Cord. mer. obs. C.D. −49° 9387
2	13 3 9.89	−19 22 19.7	Bordeaux. 1890.0 3866	18	15 6 50.90	−49 37 17.9	Arg. Gen. Cat. 20588
3	13 32 16.47	−28 11 16.2	Gen. Arg. Cat. 18520	19	15 9 54.70	−50 34 39.7	10½ mag. star mic. comp. with Star 20
4	14 29 51.02	−41 46 2.6	Berliner Jahrbuch 1911 Centauri	20	15 10 31.18	−50 43 20.2	Arg. Gen. Cat. 20675
5	14 29 26.02	−42 1 42.8	Cordoba meridian obs. C.P.D.−41°6836	21	15 12 25.70	−51 28 41.2	Arg. Gen. Cat. 20711
6	14 35 11.95	−42 53 6.1	10th magn. Star Micrometer comparison with Star 7	22	15 13 9.19	−51 26 13.9	Arg. Gen. Cat. 20726
7	14 35 20.60	−43 2 32.4	Gen. Arg. Cat. 19862	23	15 13 59.08	−51 48 12.5	Cor. Zone Cat. 15^h 766
8	14 39 1.02	−43 38 25.1	Cordoba meridian observ. C.P.D −43° 6681	24	15 20 57.75	−55 29 17.1	Arg. Gen. Cat. 20887
9	14 44 45.74	−44 46 26.3	Cordoba Zone Cat. 14h 2692	25	15 20 24.62	−55 30 53.0	10th magn. star micr. comp. Star 24
10	14 46 52.17	−45 13 13.0	Cordoba Zone Cat. 14h 2826	26	15 19 22.41	−55 54 28.2	Merid. obs. C.P.D. −55° 6550
11	14 53 18.70	−46 49 23.2	Cordoba meridian observ. C.D. −46° [illegible]	27	15 20 6.90	−56 14 26.4	Gen. Arg. Cat. 20863
12	14 56 31.20	−47 31 56.7	Cor. Zone Cat. 14^h 3435	28	15 18 54.79	−56 58 53.5	Gen. Arg. Cat. 20834
13	14 56 55.56	−47 35 42.1	Arg. Gen. Cat. 20369	29	15 19 22.53	−57 15 18.2	Cor. Zone Cat. 15^h 1116
14	15 0 38.57	−48 24 43.5	9½ mag. star mic. comp. with Star 15	30	15 14 58.10	−58 16 48.6	Cordoba meridian observ. C.P.D. −58° 5895
15	14 59 56.11	−48 32 27.1	Arg. Gen. Cat. 20450	31	15 15 12.80	−58 32 24.2	Arg. Gen. Cat. 20754
16	15 1 18.13	−48 41 56.7	Arg. Gen. Cat. 20478				

NOTES.

1911

Nov. 16. Sky good. Comet visible in the dawn more than ½ hour. Nucleus about 5th magnitude. Comet is much brighter than magnitude given in the ephemeris (probably 3d magnitude). Tail could be traced in the finder for about 2 degrees. Coma very much developed.

Dec. 27. Comet very faint owing to the proximity of the bright ($2^m.5$) comparison star, especially the first hour; star at the beginning like a disk, blurred, (15−18 seconds of arc in diameter.)

Dec. 31. Observations in α not possible. Observation was continually interrupted by clouds, measures uncertain.

1912

Jan. 1. Comet about 8th magnitude (fainter than Ephem. magnitude.) Observation was possible after many interruptions by clouds. A very diffuse but considerable tail could be traced in the finder only.

Jan. 5. Comet exceedingly faint owing to the very bad condition of the sky (very hazy); strong moonlight. Comet disappears completely at times. Bisections uncertain, but seem to agree pretty closely with the motion of the comet.

Jan. 6. Sky bad, too hazy. Moonlight interferes. Comet almost invisible, it disappears with illumination of wires. Nuclear condensation very irregular. Comet appears alternately as an object of 11-12th magnitude.

Jan. 11. Measures very irregular, clouds passing. Observation interrupted after first set in declination. Comet very faint owing to bad condition of atmosphere.

Jan. 13. Some measures both in δ and α are uncertain. Magnitude of the comet cannot be estimated owing to the moonlight and bad atmosphere.

Jan. 13. Second observation. Comet too close to a 10½ magnitude star and appears to be fainter than this star.

Jan. 16. Sky not very good. Comet seemed to be of 9th magnitude.

Jan. 17. Cloudy all night, clear in the morning only about ½ hour before observ.

Jan. 21. Images not very good, sometimes blurred.

Jan. 24. Nucleus generally well defined. Brightness about 9½.

Jan. 28. Comet about 10th magnitude, nucleus 11th magnitude or fainter. Sky good.

Jan. 29. Sky generally good, but observation often interrupted by clouds and haze. Comet 10th magnitude, nucleus 11th magnitude.

Feb. 13. Comet of about 10th magnitude, (nucleus visible). Owing to haze and clouds it was not possible to make a second set in δ.

Feb. 15. Comet 10¾-11 magnitude, nucleus 12th magnitude. Sky hazy, comet disappears sometimes.

Feb. 16. Comet 11th magnitude, with nucleus. Sky good. Comet generally invisible with illumination of the field.

Feb. 21. Comet about 10½ magnitude, nucleus visible.

Feb. 26. Sky bad, other measures impossible.

Feb. 27. Comet about 11th magnitude, with nucleus, sky hazy, air damp.

CONTENTS.

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OBSERVATIONS OF COMET *a* 1912 (*GALE*)

MADE WITH THE 11-INCH EQUATORIAL AT SMITH COLLEGE OBSERVATORY, NORTHAMPTON, MASS.

BY HARRIET W. BIGELOW.

1912 Greenwich M.T.	*	Comp.	Δα	Δδ	App. α	App. δ	log pΔ α	log pΔ δ	Red. to App. Place α	Red. to App. Place δ
h m s			m s	′ ″	h m s	° ′ ″			s	″
Oct. 4 12 19 4	1	7 0	−0 20.04	. . . .	15 26 57.05	. . . .	9.620	. . . .	+1.11	− 8.2
5 12 6 31	2	16*t* 7	−3 48.64	+6 19.7	15 29 9.76	− 0 29 33.1	9.614	0.772	+1.12	− 7.9
8 12 5 58	3	0 9	. . . .	+4 7.8	. . . .	+ 3 3 10.2	. . .	0.762	+1.07	− 7.3
14 11 52 24	5	7 7	+0 20.11	−2 2.0	15 44 50.39	+ 9 29 39.9	9.623	0.739	+1.00	− 6.1
15 11 42 23	7	8 8	−0 2.59	+1 19.1	15 46 10.15	+10 29 1.7	9.620	0.732	+0.98	− 6.2
16 11 47 0	9	7 7	−0 0.03	−4 25.3	15 47 27.70	+11 27 27.2	9.626	0.731	+0.95	− 6.3
21 11 40 36	11	8 10	+0 20.19	+2 50.8	15 53 8.32	+16 1 45.0	9.639	0.714	+0.90	− 7.2
28 11 14 46	13	9 8	+0 15.77	+1 25.2	15 59 37.67	+21 41 47.4	9.651	0.679	+0.76	− 6.0
29 11 18 1	15	20*t* 8	−0 59.07	+2 5.7	16 0 28.08	+22 27 18.7	9.656	0.680	+0.73	− 6.2
30 11 15 8	16	20*t* 9	−1 27.68	+4 56.3	16 1 18.34	+23 11 50.7	9.659	0.676	+0.72	− 6.4
Nov. 2 11 25 11	17	16*t* 7	+1 51.38	−8 43.5	16 3 46.52	+25 22 29.1	9.675	0.683	+0.65	− 7.1
5 11 28 9	18	9 8	+0 29.54	−7 50.6	16 6 12.97	+27 28 25.4	9.686	0.685	+0.60	− 7.5
6 11 31 54	19	16*t* 8	+1 22.18	−7 22.0	16 7 1.18	+28 9 37.4	9.691	0.690	+0.58	− 7.7
11 11 15 24	20	15*t* 8	+1 11.49	+3 40.2	16 11 8.37	+31 30 40.2	9.705	0.670	+0.47	− 8.5
18 11 23 48	21	10 8	−0 47.57	+0 46.2	16 17 20.19	+36 6 56.2	9.730	0.688	+0.30	−10.0
19 11 43 40	22	8 7	−0 5.90	+4 5.3	16 18 18.65	+36 46 57.6	9.731	0.717	+0.27	−10.0
20 11 40 15	24	16*t* 7	+0 54.10	+0 5.9	16 19 15.87	+37 26 18.1	9.735	0.714	+0.25	−10.6
21 11 24 28	25	15*t* 5	−2 5.35	−1 1.1	16 20 13.97	+38 5 27.5	9.741	0.692	+0.21	−10.5
30 11 30 16	26	12*t* 6	+1 0.48	−1 42.3	16 29 58.09	+44 10 3.3	9.773	0.713	−0.09	−13.4
Dec. 12 12 4 40	28	13*t* 7	−0 53.33	+6 17.9	16 46 36.65	+52 54 13.6	9.803	0.779	−0.76	−15.5

Right-ascension observations marked *t* were taken by transits; the others by micrometer measures.

Mean Places of the Comparison-Stars for the beginning of the year.

*	α	δ	Authority	*	α	δ	Authority
	h m s	° ′ ″			h m s	° ′ ″	
1	15 27 15.98	− 1 45 59.1	A.G. Nicolajew 3942	15	16 1 26.42	+22 25 19.2	A.G. Berlin B 5510
2	15 32 57.28	− 0 35 44.9	A.G. Nicolajew 3961	16	16 2 45.30	+23 7 0.8	A.G. Berlin B 5519
3	15 35 1.64	+ 2 59 9.7	Comp. with 4 Δα −0m 57s.46 Δδ +5′ 23″.8	17	16 1 54.49	+25 31 19.7	A.G. Cambr.Eng. 7480
4	15 35 59.10	+ 2 53 45.9	A.G. Albany 5260	18	16 5 42.83	+27 36 23.5	A.G. Cambr.Eng. 7501
5	15 44 29.28	+ 9 31 48.0	Comp. with 6 Δα −2m 46s.58 Δδ +1′ 43″.0	19	16 5 38.42	+28 17 7.1	A.G. Cambr.Eng. 7500
6	15 47 15.86	+ 9 30 5.0	A.G. Leipzig II 7099	20	16 9 56.11	+31 27 8.5	A.G. Leiden 5724
7	15 46 11.76	+10 27 48.8	Comp. with 8 Δα +1m 9s.30 Δδ +2′ 19″.9	21	16 18 7.76	+36 6 20.0	A.G. Lund 6718
8	15 45 2.46	+10 25 28.9	A.G. Leipzig I 5507	22	16 18 24.28	+36 43 2.3	Comp. with 23 Δα −3m 0s.99 Δδ −0′ 45″.0
9	15 47 26.78	+11 31 58.8	Comp. with 10 Δα +4m 57s.40 Δδ −4′ 15″.7	23	16 21 25.27	+36 43 47.3	A.G. Lund 6737
10	15 42 29.38	+11 36 14.5	A.G. Leipzig I 5494	24	16 18 21.52	+37 26 22.8	A.G. Lund 6719
11	15 52 47.23	+15 59 1.4	Comp. with 12 Δα +0m 24s.04 Δδ +2′ 8″.1	25	16 22 19.11	+38 6 39.1	A.G. Lund 6743
12	15 52 23.19	+15 56 53.3	A.G. Berlin A. 5702	26	16 28 57.70	+44 11 59.0	Comp with 27 Δα +2m 31s.52 Δδ −6′ 50″.7
13	15 59 21.14	+21 40 28.2	Comp. with 14 Δα −3m 30s.29 Δδ +1′ 58″.3	27	16 26 26.18	+44 18 49.7	A.G. Bonn 10557
14	16 2 51.43	+21 38 29.9	A.G. Berlin B 5520	28	16 47 30.74	+52 48 11.2	A.G. Cambr.U.S. 5089

OBSERVATIONS OF THE SATELLITE OF *NEPTUNE*,

MADE IN 1911 WITH THE 26-INCH EQUATORIAL OF THE U. S. NAVAL OBSERVATORY, WITH CORRECTIONS TO THE ELEMENTS OF THE SATELLITE'S ORBIT.

By ASAPH HALL, Professor of Mathematics, U. S. Navy, and H. E. BURTON, Assistant in the Observatory.

[Communicated by Captain J. L. Jayne, U. S. Navy, Superintendent].

As stated in the report of the Superintendent for 1911 it was not possible to begin observations of *Neptune's* satellite at the opposition of 1910-11 until January 28, 1911, because an investigation and correction of the periodic errors of the driving clock was being made, and because, also, of certain repairs which were necessary to Clark micrometer II. However, it was considered advisable that Messrs. Hall and Burton should each make a series of measures, partly with a view of obtaining some idea as to systematic differences between them.

For observations of *Neptune's* satellite and determinations of the elements of the orbit reference may be made to the following:

No. 4, Tome XLII, *Mémoires de l'Académie Impériale des Sciences de St. Pétersbourg*, VII^{E} Série; the *Astronomische Nachrichten;* the *Astronomical Journal; Monthly Notices of the Royal Astronomical Society;* and *Bulletins of the Lick Observatory.*

Referring to the often mentioned fact that many of the distance observations of *Neptune's* satellite are evidently affected with systematic errors, it may be stated that the Naval Observatory has recently purchased a Repsold filar micrometer for the 26-inch equatorial on which special attention is paid to symmetry of illumination of the wires, and to the elimination of systematic errors of observation. This micrometer will soon be fitted to the instrument.

The observations printed herewith were taken with red wires, the satellite being placed between a pair of movable threads about 6″ apart. The normal observation consisted of four position angles, four distances, four distances and four position angles. After the first set of distances the position circle was turned 180°. It was intended to use a reversing prism eye-piece for all measures, but this was possible only when the seeing was fine.

The value of the micrometer screw employed is $R = 9.''9337 + 0.''00006\ (t° - 50°F)$. The times of observation as printed are Paris times corrected for aberration. The observation equations are taken in the sense computed minus observed. The residuals are obtained by substituting in the weighted equations. The computations were made by Messrs. Eppes, Burton, and Watts, assistants in the observatory. The assumed circular elements including the mean motion, are taken from the *Connaissance des Temps.* The assumed value of the semi-major axis of the satellite's orbit is for $\log\ (\rho) = 1.47814$.

Satellite of *Neptune* 1911, Hall.

Date	Paris M.T.	Obs. p	dp	$s \sin dp$	Wt.	v	Paris M.T.	Obs. s	ds	Wt.	v
1911	h m s	° ′	° ′	″		″	h m s	″	″		″
Jan. 28	11 27 19	286 20.2	+0 23.4	+0.115	1	−0.226	11 50 2	16.78	+0.07	1	+0.146
Feb. 2	13 8 51	322 55.8	+1 55.3	+0.456	1	+0.238	13 15 59	13.51	+0.13	1	+0.234
21	13 19 15	261 1.0	+1 5.8	+0.293	0.5	−0.067	13 34 43	15.11	+0.15	0.5	+0.137
23	9 7 51	128 58.8	+1 15.0	+0.328	0.5	+0.109	.....	...	...	..	......
23	12 34 7	122 33.2	+0 50.0	+0.229	0.8	+0.053	12 51 4	16.08	−0.32	0.8	−0.112
24	11 20 25	82 55.0	+0 13.9	+0.062	1	−0.092	11 38 46	15.66	−0.31	1	−0.174
Mar. 1	10 54 19	120 30.0	+0 26.0	+0.120	1	−0.048	11 11 34	16.19	−0.26	1	−0.070
3	11 10 6	359 36.1	+1 10.0	+0.227	1	+0.073	11 26 0	11.22	−0.03	1	+0.055
4	12 24 10	295 6.1	+0 27.0	+0.128	1	−0.181	12 38 54	16.35	−0.05	1	+0.043
8	11 9 30	69 20.3	+1 26.3	+0.353	0.5	+0.145	.....	...	..	..	....
10	10 30 1	291 50.9	+1 50.2	+0.524	1	+0.211	10 49 7	16.36	+0.01	1	+0.101
16	9 58 26	288 25.0	+1 4.6	+0.310	1	−0.015	10 18 34	16.48	+0.01	1	+0.094
21	9 44 27	333 33.9	+1 57.3	+0.121	1	+0.229	10 2 41	12.47	−0.06	1	+0.034
24	10 19 42	150 46.2	−1 13.7	−0.275	1	−0.476	10 42 18	13.35	−0.45	1	−0.230
25	11 22 11	99 37.0	+0 1.2	+0.006	1	−0.154	11 42 45	16.69	−0.31	1	−0.154
28	9 23 13	279 50.3	+0 39.0	+0.186	1	−0.161	9 41 32	16.45	−0.06	1	+0.007
30	11 13 32	139 6.2	+0 27.1	+0.108	0.5	−0.051	..	...	...	..	......
31	9 22 45	96 10.2	+1 49.1	+0.517	1	+0.358	9 15 7	16.46	−0.20	1	−0.049
31	10 58 57	94 9.1	+1 4.5	+0.303	1	+0.143	11 26 19	16.29	−0.18	1	−0.031

Normal Equations.

	sin du	sin dN	sin dI	$2e$ sin Q	$2e$ cos Q	$\frac{da}{a}$	n
sin du	+2714.31	+213.57	+692.69	− 227.60	+ 517.61	+ 234.17	−48.143
sin dN		+614.68	+171.48	+ 18.32	− 201.18	− 149.76	− 9.287
sin dI			+924.51	+ 193.13	− 20.23	− 588.67	−10.163
$2e$ sin Q				+1527.51	+ 167.36	− 106.66	− 8.574
$2e$ cos Q					+2083.52	+ 85.90	− 3.424
$\frac{da}{a}$						+3587.14	−28.425

$[nn] = +\ 2.192$

Results.

sin du = + 0.01816 ±0.00293	du = + 1°.041 ±0°.168
sin dN = + 0.00982 ±0.00537	dN = + 0°.562 ±0°.308
sin dI = − 0.00186 ±0.00517	dI = − 0°.107 ±0°.296
$2e$ sin Q = + 0.00927 ±0.00332	Q = 107°.83
$2e$ cos Q = − 0.00298 ±0.00289	e = 0.00487
$\frac{da}{a}$ = + 0.00719 ±0.00226	da = + 0″.117 ±0″.037

	Sum of Squares of of Residuals	Number of Equations	*p.e.* of One Weighted Equation
sdp	0.722	19	±0″.159
ds	0.248	16	±0 .106
sdp and *ds*	0.970	35	±0″.123

1911 March 9.5 Paris M. T., corrected for aberration time.

Assumed Elements	Corrected Elements
u = 93°.775	u = 94°.816
N = 188°.281	N = 188°.843
I = 115°.853	I = 115°.746
a = 16″.272	Q = 107°.83
	e = 0.00487
	a 16″.389

Satellite of *Neptune* 1911, Burton.

Date	Paris M.T.	Obs. p	dp	s sin dp	Wt.	v	Paris M.T.	Obs. s	ds	Wt.	v
1911	h m s	° ′	° ′	″		″	h m s	″	″		
Jan. 30	12 0 50	147 45.4	+4 7.7	+0.933	0.8	+0.701	. . . .	.	. . .		
Feb. 2	14 59 35	318 52.6	+1 23.5	+0.342	1	−0.166	15 4 5	14.03	+0.07	1	+0.163
21	11 35 39	264 35.9	+0 57.1	+0.260	0.8	−0.086	12 26 18	15.47	+0.03	0.8	+0.072
23	11 43 43	125 22.4	−0 22.3	−0.101	0.8	−0.200	12 13 36	16.05	−0.40	0.8	−0.312
24	12 50 59	79 39.2	+0 26.9	+0.118	1	+0.031	13 18 9	14.81	+0.18	1	+0.127
27	11 10 59	259 8.0	+1 20.7	+0.355	1	+0.019	11 47 11	15.10	−0.11	1	−0.069
Mar. 1	12 9 54	117 48.1	+0 49.5	+0.231	1	+0.114	12 32 41	16.24	−0.11	1	−0.060
2	10 27 51	77 45.4	+4 8.1	+1.077	0.5	+0.702	10 56 55	14.87	−0.04	0.5	−0.068
4	11 2 54	295 50.8	+2 6.9	+0.594	1	+0.111	11 25 53	16.12	+0.03	1	+0.165
8	11 55 15	69 48.7	−0 51.4	−0.207	0.5	−0.220	12 21 55	13.20	+0.54	0.5	+0.313
10	11 58 48	289 59.8	+1 8.2	+0.327	1	−0.135	12 20 50	16.52	−0.03	1	+0.096
11	10 15 2	247 56.4	+1 28.5	+0.357	1	+0.063	10 33 56	13.93	−0.13	1	−0.103
16	11 33 41	285 27.5	+1 21.0	+0.390	1	−0.056	11 57 18	16.75	−0.20	1	−0.084
20	10 43 1	56 18.6	+0 11.3	+0.041	1	−0.102	11 22 3	12.01	+0.45	1	+0.311
21	11 8 17	329 9.5	+2 15.8	+0.500	1	+0.013	11 28 20	12.68	+0.07	1	+0.115
23	9 53 30	234 10.7	+0 25.0	+0.090	1	−0.162	10 17 47	12.70	−0.36	1	−0.340
24	11 54 49	146 18.1	−1 2.6	−0.240	1	−0.377	12 13 42	13.34	−0.06	1	−0.012
25	9 44 0	102 35.5	−0 11.4	−0.055	1	−0.149	10 7 30	16.59	−0.14	1	−0.111
28	11 8 45	275 25.3	+2 4.4	+0.590	1	+0.187	11 32 24	16.44	−0.17	1	−0.081
30	10 6 1	142 14.5	+0 4.9	+0.019	1	−0.117	10 26 46	13.61	−0.08	1	−0.032

Normal Equations.

	sin du	sin dN	sin dI	$2e$ sin Q	$2e$ cos Q	$\frac{da}{a}$	n
sin du	+3246.75	− 43.57	+ 485.38	+ 213.45	− 250.03	− 16.77	−64.510
sin dN		+830.12	− 53.91	+ 135.02	+ 121.29	+ 258.60	− 7.524
sin dI			+1073.21	− 35.05	− 100.77	− 823.12	−14.035
$2e$ sin Q				+1727.52	− 163.71	− 67.01	−26.624
$2e$ cos Q					+2486.25	+ 255.56	−15.176
$\frac{da}{a}$						+3862.55	−10.961

$[nn] = +3.871$

Results.

sin du = +0.01830 ±0.00303	du = +1°.048 ±0°.174
sin dN = +0.00578 ±0.00584	dN = +0°.331 ±0°.334
sin dI = +0.00974 ±0.00573	dI = +0°.558 ±0°.328
$2e$ sin Q = +0.01388 ±0.00403	Q = 58°.46
$2e$ cos Q = +0.00852 ±0.00334	e = 0.00814
$\frac{da}{a}$ = +0.00442 ±0.00295	da = +0″.072 ±0″.048

	Sum of Squares of Residuals	No. of Equations	$p.e.$ of One Weighted Equation
sdp	1.407	20	±0″.214
ds	0.557	19	±0 .140
sdp and ds	1.964	39	±0.165

1911, March 9.5 Paris M. T., corrected for aberration time.

Assumed Elements	Corrected Elements
u 93°.775	u = 94°.823
N = 188°.281	N = 188°.612
I = 115°.853	I = 116°.411
a = 16″.272	Q = 58°.46
	e = 0.00814
	a = 16″.341

OBSERVATIONS OF THE SATELLITE OF *NEPTUNE*,

MADE DURING 1911-'12 WITH THE 26-INCH EQUATORIAL OF THE U. S. NAVAL OBSERVATORY, WITH NEW ELEMENTS OF THE SATELLITE'S ORBIT.

By H. E. BURTON, Assistant in the Observatory.

[Communicated by Captain J. L. Jayne, U. S. Navy, Superintendent].

The following observations of the position angle and distance of the satellite of *Neptune*, relative to the planet, were made with the micrometer Clark II. The adopted value of one revolution of the screw was 9.″9337 + 0.″00006 (t° − 50°F).

An observation of the satellite consisted of four measures of the position angle and usually four measures of the distance, the chronometer time of each measure being recorded. The measures were made in the following order: two of the position angle, four of the distance, and two of the position angle; in two instances, however, an extra measure of the distance was made, viz.: on Jan. 13 and Mar. 10. Thus the means of the times for position angle and distance were nearly the same; the greatest difference was 6m 36s, on Dec. 18.

The position angle was measured by placing a single long wire over the center of the planet and over the satellite. The mean of the first two measures of the position angle gave the position-circle setting for the measure of the distance. The difference between this mean and the final mean of four measures was not enough to affect the measured distance. Single short wires (perpendicular to the long wire) were used for the distance measures, one being placed over the center of the planet and another

over the satellite. Coincidence was eliminated by changing the movable wire over from one object to the other after two measures were made. The wires when in use were illuminated with red light, but when the long wire was used the light was turned off of the short wires, and vice versa. The observed distances were corrected to correspond to the time of position angle and in sixty-two cases out of eighty-five the correction was zero, the largest being 0.″03.

The magnifying power used was generally 388. A power of 495 was used for the first two observations on Jan. 22, and a power of 367 was used on Dec. 18, 19, Jan. 10, Mar. 29, 31, Apr. 6, 9. On Dec. 18 a reversing prism was used. On some nights the satellite was very faint, especially on Jan. 13, Feb. 14, Feb. 28, Mar. 2, and Mar. 13.

The Paris Mean Times as published have been corrected for aberration-time. Corresponding to these times, the position angle, p, and distance, s, were computed from data given in the *Connaissance des Temps*, having assumed $e = 0$, and comparisons made with the observed values. The differences dp and ds were taken equal to the computed values minus the observed. The residuals are given under v. Under Seeing, the scale adopted is as follows: 2 = good, 3 = fair, 4 = poor. The observations of both p and s were all given the weight unity. Log (ρ) was taken equal to 1.47814. The computations were made by C. B. WATTS and checked by the observer.

Date	Paris M.T.	Obs. p	$d\,p$	$s \sin d\,p$	v	Obs. s	$d\,s$	v	Seeing
1911	h m s	° ′	° ′	″	″	″	″	″	
Dec. 18	13 34 18	245 47.7	+0 32.4	+0.124	−0.235	13.23	−0.08	+0.008	3
18	14 7 35	243 58.8	+0 49.8	+0.188	−0.166	13.16	−0.15	−0.061	3
19	14 12 8	160 23.7	−0 8.0	−0.029	−0.263	12.41	+0.11	+0.216	3
19	14 37 30	157 24.1	+1 35.8	+0.352	+0.118	12.66	−0.04	+0.067	3
1912 Jan. 10	12 42 47	273 4.0	+1 33.5	+0.441	+0.006	16.42	−0.19	−0.117	3
10	13 12 13	271 9.4	+2 34.9	+0.728	+0.295	16.11	+0.05	+0.123	3
10	14 19 5	269 25.4	+2 15.6	+0.630	+0.201	16.03	−0.06	+0.014	3
10	14 43 10	268 24.1	+2 31.8	+0.702	+0.273	15.93	−0.03	+0.045	3
13	11 21 6	93 35.4	+0 47.7	+0.225	+0.038	15.93	+0.29	+0.491	4
13	11 47 58	91 43.0	+1 51.4	+0.523	+0.340	16.81	−0.66	−0.457	4
19	11 23 2	88 35.5	+0 10.0	+0.046	−0.113	15.97	−0.26	−0.057	2
19	11 50 25	88 12.4	−0 19.9	−0.090	−0.244	15.81	−0.19	+0.015	2
19	12 39 24	86 23.8	−0 7.9	−0.036	−0.182	15.63	−0.17	+0.035	2
19	13 31 26	84 10.0	+0 21.0	+0.093	−0.044	15.42	−0.15	+0.053	2
20	12 5 22	15 24.1	+0 18.7	+0.060	+0.038	11.06	−0.01	−0.015	2
20	12 43 35	13 8.0	+0 6.2	+0.020	−0.007	11.09	−0.02	−0.033	2
20	14 1 45	7 34.4	+0 38.5	+0.125	+0.084	11.04	+0.11	+0.085	2
20	14 31 28	5 17.6	+1 1.9	+0.201	+0.153	11.28	−0.09	−0.118	2
21	11 54 33	306 2.9	+1 11.2	+0.328	−0.039	15.97	−0.15	−0.148	3
21	12 23 9	305 39.1	+0 41.0	+0.190	−0.182	15.94	−0.03	−0.024	3
22	11 9 2	266 1.8	+0 16.0	+0.072	−0.346	15.60	−0.13	−0.051	3
22	11 37 19	264 45.5	+0 35.8	+0.160	−0.256	15.19	+0.18	+0.258	3
22	12 19 5	262 9.7	+1 46.5	+0.471	+0.060	15.20	+0.01	+0.090	3
22	12 44 31	260 39.1	+2 24.6	+0.636	+0.226	15.05	+0.07	+0.150	3
24	11 39 6	123 41.8	+1 14.0	+0.345	+0.086	15.87	+0.16	+0.288	3
24	12 3 35	122 51.9	+1 18.8	+0.369	+0.111	16.09	0.00	+0.131	3
24	13 13 17	121 1.4	+1 2.7	+0.297	+0.040	16.73	−0.46	−0.324	3
24	13 43 47	120 48.8	+0 20.7	+0.098	−0.159	16.37	−0.03	+0.107	3
27	13 28 33	297 2.7	+1 55.8	+0.555	+0.148	16.79	−0.30	−0.276	3–4
27	14 7 37	297 7.2	+0 43.3	+0.209	−0.202	16.75	−0.19	−0.160	3–4
Feb. 2	12 36 52	293 41.4	+1 37.0	+0.470	+0.050	16.59	+0.08	+0.117	3
2	13 14 50	292 51.8	+1 22.0	+0.399	−0.023	16.90	−0.18	−0.140	3
5	12 32 26	112 0.1	+0 53.7	+0.262	+0.014	16.83	−0.08	+0.077	4
6	10 53 18	69 46.6	+0 37.4	+0.150	+0.078	13.88	−0.11	+0.071	3–4
6	11 20 6	69 5.9	+0 10.6	+0.042	−0.026	13.78	−0.12	+0.060	3–4
6	12 59 49	64 38.5	+0 16.5	+0.064	+0.014	13.41	−0.18	−0.012	3–4
6	13 38 56	63 7.8	−0 0.3	−0.001	−0.046	12.96	+0.10	+0.262	3–4
8	12 11 50	289 30.7	+1 26.9	+0.424	−0.006	16.77	+0.02	+0.066	3–4
8	12 47 6	288 35.1	+1 23.2	+0.407	−0.025	17.02	−0.22	−0.171	3–4
9	12 31 49	239 59.3	+2 8.6	+0.485	+0.140	13.43	−0.46	−0.372	4
9	13 21 36	237 41.3	+2 4.3	+0.462	+0.124	12.93	−0.16	−0.071	4
13	11 53 7	333 54.2	+1 43.4	+0.381	+0.190	12.70	+0.07	+0.011	3

Date	Paris M.T.	Obs. p	$d\ p$	$s \sin d\ p$	v	Obs. s	$d\ s$	v	Seeing
1912									
Feb. 13	12 26 8	332 59.2	+1 3.9	+0.240	+0.037	12.67	+0.24	+0.182	3
13	13 34 33	329 22.1	+1 31.4	+0.351	+0.129	13.06	+0.14	+0.086	3
14	10 29 18	287 18.0	+1 30.9	+0.444	+0.011	16.80	−0.01	+0.041	3
14	10 57 20	287 6.8	+0 55.1	+0.269	−0.165	16.82	−0.03	+0.022	3
14	12 4 21	284 52.1	+1 17.5	+0.378	−0.059	16.73	+0.05	+0.106	2–3
14	12 34 40	283 57.1	+1 21.7	+0.398	−0.039	16.38	+0.39	+0.449	2–3
17	10 14 30	105 59.0	+0 41.6	+0.218	−0.016	17.01	−0.24	−0.069	3
17	10 39 38	105 5.2	+0 56.3	+0.275	+0.043	16.90	−0.14	+0.033	3
17	11 32 35	103 26.5	+1 6.1	+0.322	+0.095	16.98	−0.24	−0.064	3
17	11 58 16	103 49.1	+0 0.3	+0.001	−0.224	16.89	−0.17	+0.008	3
23	9 3 35	102 46.8	+0 55.3	+0.268	+0.044	17.07	−0.38	−0.203	3
23	9 34 34	101 48.4	+1 1.5	+0.298	+0.077	16.80	−0.14	+0.040	3
23	11 54 32	97 55.1	+0 55.6	+0.267	+0.061	16.63	−0.14	+0.048	3
23	12 25 16	97 17.9	+0 39.9	+0.191	−0.012	16.69	−0.25	−0.060	3
27	12 7 19	214 29.3	+0 6.6	+0.022	−0.253	11.18	+0.07	+0.166	2
27	12 41 16	211 55.4	+0 34.4	+0.112	−0.160	11.21	−0.02	+0.076	2
28	11 58 3	135 25.6	+1 21.6	+0.345	+0.098	14.50	+0.05	+0.157	3
Mar. 2	11 11 19	313 44.4	+1 31.0	+0.389	+0.073	14.63	+0.06	+0.035	3
2	11 16 39	312 17.0	+1 42.8	+0.443	+0.120	14.86	−0.04	−0.061	3
7	11 12 1	21 1.6	+0 18.9	+0.060	+0.049	10.76	+0.19	+0.201	2
7	11 48 40	18 46.7	+0 11.8	+0.038	+0.023	10.78	+0.15	+0.154	2
10	11 24 40	193 25.8	+1 23.1	+0.264	+0.021	11.31	−0.39	−0.294	3–4
10	12 2 19	191 30.8	+0 52.2	+0.166	−0.073	11.14	−0.20	−0.103	3–4
13	10 42 8	11 50.8	+0 0.7	+0.002	−0.031	10.86	+0.07	+0.052	3–4
13	11 15 8	9 18.3	+0 26.0	+0.083	+0.045	10.93	+0.02	−0.004	3–4
16	11 14 16	182 20.2	+1 50.3	+0.354	+0.122	11.08	−0.04	+0.056	2
16	11 48 56	180 52.3	+1 8.0	+0.220	−0.011	11.22	−0.12	−0.025	2
17	11 8 51	119 38.8	+1 10.9	+0.329	+0.078	16.09	−0.12	+0.011	2
17	11 39 9	118 40.7	+1 14.7	+0.349	+0.098	16.22	−0.18	−0.047	2
18	10 58 28	77 7.5	+0 44.7	+0.188	+0.082	14.61	−0.17	+0.021	2
18	11 27 35	76 20.8	+0 26.7	+0.111	+0.012	14.36	−0.04	+0.150	2
29	10 21 50	110 48.8	+1 4.9	+0.309	+0.069	16.52	−0.15	+0.001	3–4
29	10 53 34	110 13.9	+0 46.2	+0.220	−0.019	16.52	−0.12	+0.034	3–4
31	9 13 55	345 9.4	−0 0.6	−0.002	−0.140	11.86	−0.09	−0.152	3–4
31	9 51 12	342 32.5	+0 35.6	+0.123	−0.024	11.90	0.00	−0.062	3–4
Apr. 6	9 43 22	334 3.5	+0 29.7	+0.108	−0.087	12.44	+0.06	0.000	3
6	10 15 33	332 58.9	+0 2.7	+0.010	−0.194	12.35	+0.28	+0.222	3
9	8 57 53	151 28.4	+1 2.1	+0.229	+0.001	13.34	−0.69	−0.595	4
9	9 43 45	148 50.2	+1 33.8	+0.350	+0.121	13.11	−0.27	−0.175	4
10	9 59 23	102 13.3	+0 22.0	+0.104	−0.112	16.49	−0.22	−0.046	3
10	10 32 53	101 7.4	+0 31.2	+0.147	−0.065	16.46	−0.22	−0.043	3
11	9 30 58	50 44.8	−0 36.2	−0.124	−0.135	12.35	−0.58	−0.467	3–4
11	10 6 36	48 19.5	−0 7.0	−0.024	−0.031	12.05	−0.40	−0.295	3–4

Normal Equations.

	$\sin d u$	$\sin d N$	$\sin d I$	$2e \sin Q$	$2e \cos Q$	$\frac{da}{a}$	n
$\sin du$	+15945.63	−1022.88	+1941.82	−1112.94	+ 1758.86	− 288.02	−239.232
$\sin dN$		+4230.25	− 444.64	− 287.86	− 890.77	+ 1749.92	− 75.830
$\sin dI$			+5747.51	+ 389.53	+ 200.82	− 4623.10	− 41.697
$2e \sin Q$				+8089.94	− 1885.31	− 130.12	− 71.634
$2e \cos Q$					+12370.03	− 688.88	+ 57.642
$\frac{da}{a}$						+18058.30	−114.457

$[nn] = + 11.694$

RESULTS.

$\sin du = +0.01659 \pm 0.00089$	$du = +0°.951 \pm 0°.051$
$\sin d\mathcal{N} = +0.01998 \pm 0.00171$	$d\mathcal{N} = +1°.145 \pm 0°.098$
$\sin dI = +0.00793 \pm 0.00163$	$dI = +0°.454 \pm 0°.093$
$2e \sin Q = +0.01071 \pm 0.00123$	$Q = 109°.11$
$2e \cos Q = -0.00371 \pm 0.00099$	$e = 0.00567$
$\frac{da}{a} = +0.00663 \pm 0.00992$	$da = +0''.108 \pm 0''.015$

	Sum of Squares of Residuals	No. of Equations	Probable Error of One Equation
$s \sin dp$	+1.504	85	±0″.093
ds	+2.641	85	±0 .123
$s \sin dp$ and ds	+4.145	170	±0″.107

1912, February 15.5 Paris Mean Time, corrected for aberration-time.

Assumed Elements	Corrected Elements
$u = 225°.090$	$u = 226°.041$
$\mathcal{N} = 188°.423$	$\mathcal{N} = 189°.568$
$I = 115°.698$	$I = 116°.152$
$a = 16''.272$	$Q = 109°.11$
	$e = 0.00567$
	$a = 16''.380$

COMET *b* 1913 (*METCALF*).

A new comet was discovered at South Hero, Vermont, by METCALF on Sept. 1.56 G.M.T. at R.A. = 6^h 50^m; Decl. = +57°.

The elements as computed by Professor CRAWFORD and Miss LEVY at Berkeley, Cal., from observations taken on Sept. 2, 19, and 16 are:

T = 1913, Sept. 13.97 G.M.T.
$\omega = 117°\ 26'$
$\Omega = 157\ \ 16$
$i = 143\ \ 24$
$q = 1.358$

METCALF'S COMET.

F. E. SEAGRAVE.

Greenwich Midnight 1913	α h m s	δ ° ′ ″	Log r	Log Δ
Oct. 1	23 4 34	+71 32 44	0.12554	9.77924
5	21 51 25	+59 36 6	0.12960	9.75285
9	21 17 56	+45 31 32	0.13442	9.75411
13	21 0 25	+32 0 38	0.13994	9.78292
17	20 50 20	+20 34 55	0.14608	9.83020
21	20 44 18	+11 34 27	0.15278	9.88555

This ephemeris of METCALF's Comet is based upon elements of CRAWFORD and LEVY in *Harvard Bulletin*, 524. If these elements are right, METCALF's comet will be nearest the Earth October 7, distance 52,500,000 miles.

COMET *d* 1913 (*DELAVAN*).

Announcement has been made of the discovery of a comet by DELAVAN at La Plata, Argentina, on Sept. 26.5978 at R.A. = 21^h 55^m $18^s.4$, Decl. = −2° 34′ 27″. It is visible in a small telescope. He states it may possibly be WESTPHAL's comet.

An observation obtained by BURTON at the U. S. Naval Observatory gives its position on Sept. 27.7204 as R.A. = 21^h 50^m $37^s.6$, $\delta = -1°\ 37'\ 33''$.

LEUSCHNER telegraphs that NICHOLSON and Miss LEVY find DELAVAN comet identical with WESTPHAL's, by interpolation applied to ephemeris in *Astronomische Nachrichten* No. 4619. The period is 61.121 years.

COMET *c* 1913 (NEUJMIN).

On Sept. 6, 1913, the discovery of a new asteroid by NEUJMIN was announced. On Sept. 7, BACKLUND telegraphed from Pulkowa that the supposed asteroid was a comet, and GRAFF telegraphed from Bergedorf that the Pulkowa object is a comet with a short tail.

AITKEN saw no trace of nebulosity in the moonlight, but BARNARD describes the object as being of 11.5 magnitude, with a stellar nucleus followed by a faint coma. The object is tentatively given a place among the 1913 comets.

The following are the elements and ephemeris by S. EINARSSON and S. B. NICHOLSON computed at Berkeley, Cal. from observations made on Sept. 6, 9, and 13.

ELEMENTS.

T = 1913, Aug. 15.95 G.M.T.
$\omega = 346°\ 4'$
$\Omega = 348\ \ 3$
$i = \ \ 15\ \ 28$
$q = 1.552$
$e = 0.8211$
P = 25.55 years.

CONSTANTS FOR THE EQUATOR 1913.0.

$$x = r\,[9.999337]\sin\,(\ 64°\ 31'\ 57'' + v)$$
$$y = r\,[9.893357]\sin\,(332\ \ 00\ \ 29\ \ + v)$$
$$z = r\,[9.796146]\sin\,(338\ \ 29\ \ 23\ \ + v)$$

EPHEMERIS FOR GREENWICH MEAN MIDNIGHT.

1913	h m s	° ′ ″	log Δ	Mag.	Br.
Oct. 2.5	23 34 58.4	+ 9 32 50			
4.5	34 15.6	10 06 56	9.8382	11.98	0.64
6.5	33 39.7	10 39 18			
8.5	33 11.1	11 10 00	9.8560	12.09	0.58
10.5	32 50.0	11 39 06			
12.5	32 36.5	12 06 40	9.8748	12.21	0.52
14.5	32 30.5	12 32 45			
16.5	32 32.4	12 57 26	9.8945	12.32	0.47
18.5	32 42.0	13 20 50			
20.5	32 59.6	+13 43 05	9.9148	12.44	0.42

NOTICE.

It has been suggested that the *Astronomical Journal* compile a general index of its first 25 volumes. Such a publication would involve considerable expense, and though such expense would be partially defrayed by the sale of the index, we would hesitate to take any steps before ascertaining how far the desire for such a publication extends. We would therefore gratefully receive any comments or suggestions.

CONTENTS

ACTING EDITOR BENJAMIN BOSS, ALBANY, N.Y.; ASSOCIATE EDITORS SETH C. CHANDLER AND GEORGE W. HILL. DIRECTORS OF THE GOULD FUND OF THE NATIONAL ACADEMY OF SCIENCES AND PROF. ERNEST W. BROWN, OF YALE UNIVERSITY. PUBLISHED BY THE DUDLEY OBSERVATORY, ALBANY, N.Y., U.S.A., TO WHICH ALL COMMUNICATIONS SHOULD BE ADDRESSED. PRICE, $5.00 THE VOLUME. PRESS OF THOS. P. NICHOLS & SON CO., LYNN, MASS. *Closed Sept* 29, 1913.

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PROPER MOTIONS OF TELESCOPIC STARS,

By GEORGE C. COMSTOCK.

Vol. XII, Publications of the Washburn Observatory, contains an investigation of the proper motions of two hundred and twenty-two faint stars, which is based upon a discussion of micrometer measures covering an average interval of about half a century. During the five years that have elapsed since the publication of that volume, its material has been revised and extended so that there is now available for a study of the motions of the fainter stars a body of data considerably greater than was then extant, viz.: I have measured with the micrometer and have discussed the motions of five hundred and thirteen stars fainter than the eighth magnitude. Of these stars, three hundred and ninety are believed to show proper motion in the strict sense of that term. The motions of the remaining one hundred and twenty-three stars are either demonstrably orbital or are of uncertain character. The later determined proper motions are naturally somewhat inferior in precision to those of Vol. XII, since the most promising material was first taken in hand. An examination of the precision to be attributed to the data as a whole is given hereinafter.

The large proportion of sensible proper motions thus found among telescopic stars has produced some surprise and incredulity. It has indeed been denied by certain astronomers that any of these motions are other than orbital or the result of error in, or misinterpretation of, the data employed; *e.g.* BURNHAM, Measures of Proper Motion Stars, Carnegie Institution, 1913. "There is at this time no relative evidence of any character that any really faint star, not associated and moving with a larger star, has any sensible proper motion at all;" and INNES, Circular No. 6 of the Union Observatory, "The series of measures connecting bright and faint stars . . . may be adverted to because they prove in the main the almost absolute fixity of very faint stars." To a similar effect, see AITKIN, Publications *A. S. P.*, August, 1913, p. 220. In the conviction that such opinions are erroneous and that a large part, probably a major part, of the stars brighter than the thirteenth magnitude possess proper motions that are readily measurable, it is the purpose of this paper to set forth some of the results obtained from a discussion of the measures of the stars above named and certain others supposed to be comparable with them in respect of proper motions.

It is here premised that the methods of observation and discussion employed with this material have been substantially unaltered from those set forth in the Introduction to Volume XII, cited above, but in two respects important changes have been introduced, viz.:

A. MAGNITUDES. The magnitudes assigned to the stars in Vol. XII were based upon eye estimates made in connection with the micrometer observations, chiefly by W. STRUVE, OTTO STRUVE, DEMBOWSKI, and myself. Through the kindness of Professor E. C. PICKERING, a considerable number of these stars have been photometrically observed at the Harvard Observatory while others are found to have been included in the routine work of that institution. In so far as they are available, these Harvard magnitudes are now substituted in place of those hitherto employed and a systematic correction of +0.4 m. has been applied to the remaining estimated magnitudes in order to reduce all to a uniform system. From one hundred and thirty-three stars observed at Harvard, I find

$$\text{Harvard} - \text{Estimated} = +0.41 \pm 0.037,$$

and this difference is practically constant over the range of magnitudes here in question, *i.e.* 8m to 12m.

B. PROPER MOTIONS. In the volume above cited, the proper motions of the comparison stars were either taken directly from NEWCOMB's Fundamental Catalogue or were reduced to that standard by the use of systematic corrections. I now adopt the proper motions of Boss, taken from the *Preliminary General Catalogue*, whenever that is available. In many cases, however, I have found it necessary to derive these proper motions from original sources, using Boss's weights and systematic corrections. A large influence has been exerted upon these determinations by observations especially made for this purpose by

Mr. A. S. Flint, using the Madison meridian circle. In a few cases, I have adopted Newcomb's proper motions reduced to the Boss system by application of the corrections given in *The Astronomical Journal*, No. 531.

Precision of the Proper Motions.

Since the motions of the brighter comparison stars and those of the fainter stars dependent upon them, have in every case been derived through least square solutions, the material is available for determining the probable error of each concluded proper motion. This probable error has been derived for each co-ordinate of every star employed, and, through a comparison of these probable errors with those given in the *P. G. C.*, a relative measure is established for the precision attained in the concluded motions of the faint stars, at least in so far as that precision relates to accidental error. This comparison with Boss has been made as follows:

From the first two stars fainter than magnitude 7.0 contained in each hour of right ascension of the *P. G. C.* I find as the average probable error of a centennial proper motion of these stars in each co-ordinate, the numbers given under the heading $r(\mu)$ in the first line of the following Table I. By an entirely similar process, I find the corresponding probable errors of the centennial proper motions of the faint stars micrometrically observed. These stars, about four hundred in number, will hereinafter be represented collectively by the symbol *C*. For most of the following purposes, there will also be included within the designation *C* the proper motions of one hundred and twenty-eight comparison stars fainter than the seventh magnitude, but these are not included in the discussions of probable error or systematic error. The probable errors of the adopted proper motions of the comparison stars are in every case included in the probable errors of the concluded proper motions of the *C* stars.

Table I.

Average Probable Error of a Centennial Proper Motion.

Source	Limits of Mag.	Mean Mag.	$r(\mu)$ R.A.	$r(\mu)$ Dec.	No. of Stars
			″	″	
P. G. C.	7.0 –	7.3	±0.62	±0.54	48
C.	7.0 – 10.0	8.6	0.71	0.57	48
C.	10.0 – 13.	11.1	0.73	0.62	48

It may be noted as a matter of passing interest that in each of the six cases given in this table a larger value of $r(\mu)$ results from discussion of the stars between 0^h and 12^h of right ascension than is found between 12^h and 24^h. A more important consequence is, that in the extension of Boss's system of proper motions to much fainter stars a measure of precision has been attained that is fairly comparable with that of the best determined stars of the seventh magnitude, in so far, at least, as accidental error is concerned.

It is of course possible that in the results of a method not largely tried and not subject to any independent control, there may be systematic errors present of sufficient magnitude seriously to affect the conclusions to be drawn from the data. The presence of such systematic error in the results here considered has indeed been affirmed on the ground that the stars display a marked southerly drift in declination. This drift is assumed, by Lau, to arise from some systematic error in the micrometer observations. While the existence of such an error of appreciable magnitude, *e.g.* 0″.1 and of the form assumed by Lau, seems to me intrinsically improbable, I have nevertheless sought to control the character of these proper motions by comparing them with others derived in the conventional manner from meridian circle observations, and therefore presumably free from the alleged error peculiar to observations with a filar micrometer.

The accompanying Table II, presents such a comparison of all cases in which Boss's *P.G.C.*, contains the proper motion of a *C* star. It is of course limited to the brighter stars, since the *P.G.C.* scarcely reaches beyond the eighth

Table II.

Difference of Centennial Proper Motions. C.–B.

Stars R. A.	Stars Dec.	Stars Mag.	In R.A. A′ C	B	C–B	p. e.	In Dec. D′ B	C	C–B	p. e.
h m	°		″	″	″	″	″	″	″	″
6 58	+20	8	−11.2	−10.9	−0.3	±0.69	+2.8	+4.3	−1.5	±0.88
9 22	+46	8	− 0.9	− 1.2	+0.3	±0.75	−4.8	−5.3	+0.5	±1.07
13 10	−11	8	+ 2.0	+ 0.9	+1.1	±1.08	−2.4	−0.7	−1.7	±0.89
17 11	+25	8	−10.4	−10.4	0.0	±0.3	−0.1	−0.3	+0.2	±0.3
20 6	+21	7	+ 0.1	− 0.4	+0.5	±0.61	−0.7	−1.0	+0.3	±0.62
21 40	+28	7	− 1.1	− 1.1	0.0	±0.66	−5.5	−6.0	+0.5	±0.56
22 32	+40	7	− 1.1	− 0.7	−0.4	±0.96	−1.3	−1.1	−0.2	±0.76
23 43	+16	8-8	+ 8.0	+ 7.5	+0.5	±1.40	−7.3	−6.4	−0.9	±1.05

magnitude. The probable errors of the several determinations being known, I have formed from them the probable error of each difference COMSTOCK–BOSS.

The last line of this table represents a star not in the *P. G. C.* It is one of the comparison stars whose proper motion I have derived from the star catalogues using Boss's systematic corrections and weights and for which a proper motion has also been found micrometrically. The comparison is between the two proper motions thus obtained.

While the data here furnished are very limited, they suffice to show that in thirteen out of sixteen cases the actual difference, C–B, is less than the probable error of its determination. The mean values of this difference,

In R.A., $C\text{-}B = +0''.17 \pm 0''.32$. In Dec. $C\text{-}B = -0''.35 \pm 0''.28$.

lend little support to the claim of an appreciable systematic difference between the proper motions C and B. They admit, however, the supposition of a small southerly drift in C not shown by B and to test further this possibility, I have divided that part of the sky within which my observations are comprised, into seven regions of approximately equal area and within each region, I have formed the mean value and the median value of the centennial proper motion in declination for all stars included between the magnitudes 7.0 and 9.0. From Boss' *P.G.C.*, I have formed similar quantities based upon all stars there designated as fainter than the magnitude 7.0 and a comparison of these quantities is contained in the following Table III. The individual areas P, 2, 4, etc., are hereinafter defined in connection with the discussion of the systematic motions of the faint stars.

TABLE III.

CENTENNIAL MOTION IN DECLINATION.

Region	Comstock Stars	Comstock Mean	Comstock Median	Comstock Q	Boss Stars	Boss Mean	Boss Median	Boss Q
		$''$	$''$			$''$	$''$	
P	13	+0.3	+0.4	2.2	35	− 0.8	−0.7	0.7
2	10	−1.4	−0.4	0.8	34	− 7.3	−3.8	0.3
6	14	−1.7	−2.4	0.3	13	− 3.9	−1.2	0.4
10	8	−2.5	−2.3	0.3	11	−10.2	−3.1	0.1
14	7	−1.6	−1.4	0.4	10	− 4.4	−3.0	0.4
18	9	−0.6	−0.1	0.8	13	− 6.3	−0.3	0.5
22	15	−1.0	−0.7	0.4	17	− 0.1	−1.3	0.5
	76				133			

An inspection of this table suffices to show that there is in fact a marked southerly drift shown by the stars of both categories and numerically it is of greater amount in the B stars than in C. A partial explanation of this numerical excess lies in the fact that on the average the B stars are about a magnitude brighter than the C's. They are also in considerable part, stars to which attention has been attracted by the fact of large proper motion while the C stars are more nearly typical of all stars of like magnitude.

The quantities designated in the table by the symbol Q are the quotients obtained by dividing, in each area, the number of stars moving north by the number moving south and the generally smaller values of Q associated with the Boss stars indicate that the percentage of southerly moving stars is certainly not greater in C than in B. Combining these results with those above derived from the comparison C-B, I conclude that any systematic difference that may here exist is so small as to be, for the present, of negligible consequence.

Disregarding, now, all questions of probable error or precision in the motions attributed to the C stars, let it be assumed that these motions are largely or wholly fictitious, *e.g.*, due to erroneously assumed proper motions of the comparison stars. We should expect to find in this case a chaotic body of small apparent motions. Any trace of law or order manifested by these motions when taken collectively is in some measure opposed to the assumption above made, and, if these motions be found to show in large measure those systematic features that are known to exist among the brighter stars, there will be created at least a *prima facie* case against the unfavorable supposition above made. I proceed to consider some of these features.

The well marked southerly streaming tendency of the faint stars, above attested by both B and C, is easily surmised to be a consequence of the solar motion through space, since that motion has a large component directed toward the north. Since determinations of that motion have been based, hitherto, almost wholly upon stars brighter than the eighth magnitude, it is a matter of intrinsic interest to seek such a determination from fainter stars, *e.g.*, those here in question. The problem is considerably complicated by the recognized tendency of the brighter stars to flow in certain determinate directions and there is no evidence extant to show in what measure this tendency is shared by the fainter stars. Does the two stream hypothesis of KAPTEYN, or the alternative unitary theory of SCHWARZSCHILD, represent a small eddy localized in one part of celestial space or does it extend more widely, *e.g.*, to the region of the twelfth magnitude stars? Seeking a test of the genuineness of the proper motion attributed to the C stars, I have applied them to the above problems, using the second method of SCHWARZSCHILD, *Göttingen Nachrichten*, 1908, p. 191, and *Astr. Nachr.* No. 4291. This method is especially advantageous in the present case, because of the facility of its applica-

tion to scanty data and the comparatively small scope that it affords to the computer's prepossessions. Its application has been as follows:

Within a given area of the sky there is determined by actual enumeration the relative number of proper motions, pointing in determinate directions, *e.g.*, between 0° and 10° of position angle; 10°–20°; 20°–30°; etc. From these numbers, there are determined by summation certain others, l, which give rise to two functions represented in SCHWARZSCHILD's notation, by the second members of the following equations:

$$L = l_1+l_2-l_3-l_4 \qquad\qquad K = l_1l_3-l_2l_4$$

which contain the solution of the problem. Of these L is a single period and K a double period function of the position angle p, reckoned from 0° to 360°, such that the value of p for which $L = 0$, $p = \theta_1$, denotes the direction of the apex of solar motion, and the values of p for which $K = 0$, $p = \theta_0$, fix the directions of star drift.

It will appear from the above that the positions of apex and vertex are to be determined by means of their directions from certain areas within which proper motions are grouped and enumerated. It is therefore a matter of importance to define the exact point within each area to which the proper motion and the resulting θ_0, θ_1, shall be assumed to relate. Further, it is desirable, if possible, to reduce each observed proper motion to this assumed origin. That part of the sky within which my data lie has been divided into seven regions of approximately equal area, which are designated by the following symbols, viz.: P represents that part of the sky lying north of the parallel of declination 45°. The assumed origin for this area is the pole. 2 represents that part of the sky south of declination 45° lying between 0^h and 4^h of right ascension. The assumed origin has the co-ordinates $\alpha = 2^h$, $\delta = 0°$. 6 represents the similar area included between 4^h and 8^h of right ascension, with origin at $\alpha=6^h$, $\delta=0°$. The remaining areas, 10, 14, 18, 22, are similarly defined. In place of each observed proper motion, I have sought to substitute the motion of a star situated at the origin, and moving along a right line parallel to the path of the real star. To determine the apparent direction (position angle) of motion along the line thus defined, there is required, in general, a knowledge of the proper motion, radial velocity and parallax of the observed star, and since the last two items are wholly unknown, recourse must be had to hypothesis, viz: that the mean radial velocity of the stars is zero, and that this mean value may be substituted for the actual value in each case without causing serious systematic error in the final result. I shall not here discuss the legitimacy of this assumption, but adopt it provisionally as the best available approximation to the truth. When the radial velocity is zero, the effect of parallax disappears from the problem and we have the following formulæ by which to pass from the observed proper motion to the equivalent motion of a star situated at the origin of the given area. Let A' and D' denote the observed centennial proper motions expressed in seconds of arc of a great circle; α and δ the equatorial co-ordinates of the star; α_0 the right ascension of the assumed origin; p the position angle of the reduced proper motion:

Then

For the area P,

$$\tan\,\theta = {}^{A'}\!/_{D'}\sin\delta \qquad\qquad p = 180° + \alpha - \theta$$

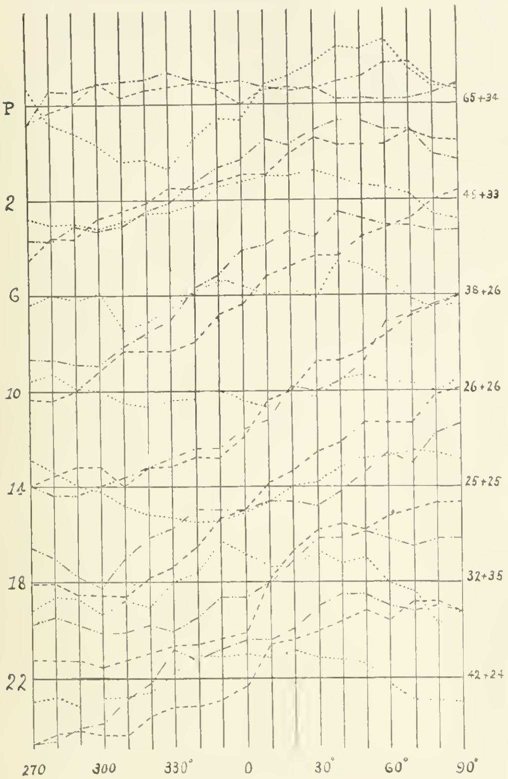
For the other areas

$$\tan p = \frac{A'\cos(\alpha_0-\alpha)}{D'\cos\delta} + \tan\delta\sin(\alpha_0-\alpha)$$

By means of these formulæ, and auxiliary tables based upon them, I have obtained the value of p corresponding to each proper motion employed and these values constitute the basis of the following results. I have rejected from the data to be employed for the present purpose every star for which the probable error of the direction of the observed proper motion exceeds 45°. This criterion excludes principally those stars whose proper motion is exceptionally small and to some extent constitutes a preference which impairs the representative character of the data. Quantitatively, however, this effect is small since only 5 per cent. of the data otherwise available, has been thus rejected.

My original purpose was to divide the data into two parts, stars brighter than, and stars fainter than the magnitude 10.0, and to seek from each of these parts an independent determination of the apex and vertex of stellar motions. This has been done, but it appears in the solution that the preferential stellar motion that determines the vertex is much less pronounced in its effect than is the solar motion and I therefore present here two determinations of the apex, made as above indicated, but only a single solution for the vertex. These solutious are based upon the total data available, viz. three hundred and fifty-three stars whose proper motions depend upon micrometer measures and one hundred and twenty-six stars fainter than the seventh magnitude, which have been used as comparison stars and whose proper motions are in a few cases taken from the *P. G. C.* (Boss) but have for the most part been derived by myself.

The values of L and K, upon which the solutions depend, are presented graphically (without any adjustment or smoothing) in the accompanying plate in which abscissæ represent position angles and ordinates the corresponding values of L and K. Only that part of the curve lying between position angles 270° and 90° is here shown, since from the nature of the data, the part lying



L, ** 7m to 10m L, ** 10m to 13m K, ** 7m to 13m

between 90° and 270° is a mere repetition of the first half; with altered sign in the case of L. The symbols printed at the left of the plate, represent the regions to which the several curves relate. The numbers printed at the right of each region represent respectively the number of stars brighter than and fainter than the tenth magnitude, whose proper motions enter into the determination of the L curves. Their sum gives the corresponding number upon which the K curve is based. All of the L curves are comparable one with another in respect of amplitude, since the numbers furnished directly by the data have been multiplied by a factor containing the number of stars, n, as a divisor. Similarly the K curves have been made comparable through multiplication by a factor containing the inverse square of n*. A smooth curve, not here reproduced, has been made to conform as nearly as may be to the plotted lines of the figure and the points at which this curve cuts the x-axis, furnish the values of θ_1 (apex), and θ_0 (vertex) given in the following Table IV. There are included in this table, values of certain other functions β/h, β/a, for which see Schwarzschild, *loc. cit.*

* Through an error of draughting, discovered too late for remedy, the ordinates of the K curve for the area 10 are given in the plate only one-half of their true value. This error has no effect upon the value of θ_0 furnished by the curve.

Table IV.

Data for the Determination of Apex and Vertex.

Stars 7.0 – 10.0. Magnitude.

Region	P	2	6	10	14	18	22
**	65	49	38	26	25	32	42
Mean Mag.	8.6	8.4	8.7	8.8	8.7	8.6	8.4
θ_1	277°	330°	339°	25°	38°	4°	325°
β/h	4.8	1.7	1.7	1.1	1.4	1.6	1.5

Stars 10.0 – 13. Magnitude.

Region	P	2	6	10	14	18	22
**	34	33	26	26	25	35	24
Mean Mag.	11.0	10.9	11.3	11.0	11.3	11.2	11.0
θ_1	287°	325°	5°	21°	12°	11°	358°
β/h	3.0	1.8	1.4	1.3	0.9	1.0	1.1

Stars 7.0 – 13. Magnitude.

Region	P	2	6	10	14	18	22
**	99	82	64	52	50	131	66
Mean Mag.	9.4	9.4	9.7	9.9	10.0	9.8	9.3
θ_0	272°	250°	...	124°	112°	153°	236°
β/a	0.66	0.69	...	0.73	0.67	0.73	0.59

An inspection of the plate will show that in the region P, both curves representing the function L are very flat (small amplitudes), and that their intersections with the x-axis and the values of θ_1 dependent upon this intersection are ill determined. The value of β/h is also abnormally great. Corresponding to this condition, we find from all the data that the position of the apex furnished both by the brighter and by the fainter stars taken separately, falls within the area P and accounts for the small effect of the solar motion there manifest. A similar condition with respect to K obtains in the areas 6 and 18 (position of the vertex), where it has been necessary to unite the two areas in order to obtain any adequate representation of the curve. The resultant curve is plotted for the area 18. The curve for the area 6, although shown in the plate is omitted from the following discussion. Since the origins of these areas, 6 and 18, are at diametrically opposite points of the sky, it is evident that they must present a similiar distribution of proper motions if the actual motions of the stars in the two areas are not essentially unlike, viz.: we should expect to find satified for both θ_0 and θ_1 the purely geometrical relation,

$$\theta_6 + \theta_{18} = 360°$$

with similar relations for every pair of opposite areas. The extent to which this *a priori* condition is satisfied by the data in hand, is shown in Table V, which contains the sums of corresponding θ's as given in Table IV. The value of θ_1 for all stars is a weighted mean of the separate results furnished by the stars brighter and fainter than the tenth magnitude, the weights being made proportional to the number of stars entering into each determination.

Table V.

Similarity of Stellar Motions on Opposite Sides of the Earth.

Regions	θ_1 Bright	θ_1 Faint	θ_1 All	θ_0 All
2+14	368°	337°	358°	362°
6+18	343	376	358	. .
10+22	350	379	362	360

The discordances from the theoretical value of 360° are here so small that I regard it as legitimate to assume a similar distribution of proper motions upon opposite sides of the Earth, and to unite in a single result for the K curve the data furnished by the regions 18 and 6, the position angles in the latter case being subtracted from 360°.

By the methods of Schwarzschild, I have derived from the data of Table IV, co-ordinates of the apex of solar motion and of the vertex of stellar motions as is shown in Tables VI and VII. Unit weight was assigned to each value of θ employed. These tables also contain the results of all other investigations of these motions that are known to me as having been made by like methods.

Table VI.

Determinations of the Apex.

Ref. No.	**	Mag.	A	D	λ	β	$\sqrt{2}\frac{\mu}{B}$
1	2375	5.	268°	+26°	19°	22°	1.24
2	3934	7.	266	+33	25	27	0.99
3	1327	8.	281	+36	33	15	1.92
4	276	8.6	280	+58	55	23	1.00
5	203	11.1	288	+71	69	23	1.22

Table VII.

Determinations of the Vertex.

Ref. No.	**	Mag.	A	D	λ	β	B/A
1	2375	5.	276°	− 7°	352°	+ 2°	0.56
2	3934	7.	273	− 6	351	+ 3	0.67
3	1327	8.	266	−24	332	0	0.47
4 & 5	479	9.7	277	−28	334	−10	0.62

The reference numbers in Tables VI and VII have the following meanings:

1. Rudolph's discussion of the Bradley stars. *Ast. Nach.* No. 4369.

2. Schwarzschild's application of his first method to the Groombridge catalogue. *Göttingen Nachrichten*, 1907, p. 614.

3. Beljawski's treatment of the Porter Catalogue of Proper Motion Stars, *Astr. Nach.* No. 4291.

4-5. The brighter and fainter stars, C, discussed in this paper.

The stellar magnitudes assigned to Nos. 1, 2, and 3, as mean values are only crude estimates, but, presumably, they will suffice to arrange the several determinations in the order of the average brightness of the stars upon which they depend.

The precision with which apex and vertex are here determined from the C stars, Nos. 4 and 5, may be inferred in some measure from Table VIII. Each observed value of θ_1 or θ_0 represents a great circle drawn from the origin in question toward apex and vertex respectively, and in the ideally perfect case these several great circles should meet in a common point. I have computed the angular distance of the adopted apex and vertex from each great circle used in its determination and these distances are shown as residual errors in Table VIII.

Table VIII.

Residual Errors.

Region	Apex from Stars 7.0–10.0 Mag.	Apex from Stars 10.0–13. Mag.	Vertex 7 Mag. – 13 Mag.
P	3°	1°	5°
2	1	16	9
6	12	11	. .
10	0	8	0
14	7	6	7
18	2	4	6
22	9	11	0

If the twenty-five or fifty or one hundred proper motions included within each area were in any large measure illusory or fictitious, they might through accident indicate a point of convergence that could be recognized as the apex or vertex of stellar motions. It seems hardly within the limits of credibility that the several areas should by chance agree upon the position of apex and vertex in any thing like the measure shown in Table VIII. It is even less credible that their fictitious determination of apex and vertex should be in even an approximate agreement with the results of other determinations.

To bring out more clearly the nature of the discordances among the results shown in Table VI, there are given in that table under the headings λ, β, the galactic co-ordinates of the several positions assigned to the apex. It is ap-

parent here that the discordance is wholly in galactic longitude, the latitudes being in good agreement. The progressive increase in the value of λ with diminishing brightness of the stars from which it is determined was pointed out by me seven years ago as indicated by the material then available, viz., before any of the discussions here utilized had appeared. (*Publications of the Astronomical and Astrophysical Society of America*, Vol. I, p. 270). See also B. Boss, *Astr. Jour.*, 649–650, p. 15, where a like result is reached from other data. A somewhat similar case is presented by the determinations of the vertex where stars of diminishing brightness seem to furnish approximately the same galactic latitude, associated with a progressively diminishing longitude of the direction of preferential motion. While it is possible to regard these varying longitudes as the result of accidental error, it seems at least equally plausible to consider them as a real effect of systematic differences between the motions of the nearer and the remoter stars, *e.g.*, the influence of a group of stars relatively near the Earth and possessing a common motion in the plane of the galaxy.

The last column in Table VI represents under the rubric $\sqrt{2}\frac{\mu}{B}$ the concluded ratio of the Sun's linear velocity to the mean component of stellar velocity, perpendicular to the line of preferential motion. The quantity B/A in the last column of Table VII gives the concluded ratio of the components of stellar velocity perpendicular to, and parallel to, the line of preferential motion, *i.e.*, it is a numerical measure of the amount of preference for motion along the line of vertices. The general agreement of these quantities among themselves, with possible exception of No. 3, Table VI, is eminently satisfactory. It apparently indicates that the preferential motion is about equally manifest in all stars down to the twelfth or thirteenth magnitude and that the linear velocities of the telescopic stars are not markedly different from those of the lucid stars. This conclusion is distinctly an unexpected one, since, *a priori*, it would seem probable that the fainter stars are on the whole of smaller mass and greater velocity than the larger ones. The observed fact is, however, in agreement with Campbell's latest conclusion drawn from radial velocities. *L. O. Bulletin*, No. 196.

The investigations summarized in Tables VI and VII are the only ones known to me as having been executed by the method here adopted. Their inter-comparison seems more legitimate than the comparison of any of them with results otherwise obtained, although they are not entirely homogeneous in character among themselves; *i. e.* it is not apparent that they depend upon identical values of the precession constant. I have used the precession constant adopted by Boss for the *P. G. C.*, without attempting to correct it. It appears indeed from the data that no sensible correction is required since, of the stars lying outside the polar zone, P, 203 give negative values of the proper motion in right ascension, 207 give positive values, and 3 values are zero.

The foregoing discussion of the apex and vertex of stellar motions has been based wholly upon the directions in which the observed proper motions point, and all consideration of the amount of the proper motions has been omitted. We may therefore appeal to the average amount of the observed proper motions as a further criterion of their genuineness. Although the opinion seems to be widely held that for stars fainter than the ninth magnitude, the proper motions are generally inappreciable, evidence to the contrary has long been available. A cogent presentation of this evidence was made twenty years ago by Auwers, who showed in the Introduction to the Berlin volume of the *A. G. Zones*, $+15°$ to $+20°$ *Declination*, p. (141), that from the second to the seventh or eighth magnitude, the mean value of the product, stellar magnitude $\times$ proper motion is nearly constant, *i. e.*, $m\mu = C$. Extrapolating this relation, Auwers there estimates the mean proper motion of the tenth magnitude stars at 4″ per century. Surely an appreciable quantity. I have employed the material here available as a control upon Auwers' extrapolation and I give below the values of $m\mu$ furnished by the C stars, when classified by magnitudes. There is also given, in somewhat condensed form, the data upon which Auwers' conclusions were based. Minute precision seems to be unnecessary here, since neither the proper motions nor the magnitudes used by Auwers can be assumed exactly to conform to the Boss-Harvard system here employed.

Table IX.

Relation of Proper Motion to Magnitude.

Mag.	No. of Stars	Mean $m\mu$	% Rejected	Authority
2.3	74	26″	6	Auwers
3.4	175	31	11	
4.0	111	32	11	
4.3	106	33	8	
4.7	166	30	8	
5.1	450	32	7	
5.6	187	31	8	
5.8	242	37	8	
6.0	531	33	6	
6.3	452	35	6	
6.8	300	37	6	
7.4	172	37	9	
7.5	43	40	14	Comstock
8.5	99	42	12	
9.5	114	32	3	
10.5	108	32	5	
11.5	71	34	4	
12.5	41	37	0	

The first column of the above table gives the mean stellar magnitude; the second, the number of stars included in this mean; the third the value of the product of mean magnitude and mean centennial proper motion found by AUWERS and the mean value of magnitude multiplied by proper motion for COMSTOCK. In forming mean values, a certain number of stars with abnormally large proper motions were rejected and the number of such rejections for each group of stars is shown in the fourth column as a percentage of the total data available. The criterion by which proper motions are to be rejected as "abnormally large" is somewhat arbitrary and for the sake of uniformity, I have followed that adopted by AUWERS, viz. a value of $m\mu$ exceeding $120''$. While the individual lines of the table present considerable irregularity in the number of stars rejected, in the mean the lucid and telescopic stars agree in showing about seven per cent of proper motions designated as abnormal by the criterion here adopted.

AUWERS interprets his data as indicating a progressive increase in the values of $m\mu$ with diminishing brightness of the stars and this view is not inconsistent with the supplementary data presented above, provided this product be assumed to reach a maximum at about the eighth magnitude. It is perhaps equally proper to regard the product $m\mu$ as constant, at least from the fifth to the twelfth magnitude, and to assign it a mean value of $35'' \pm 1''$. The larger values of $m\mu$ associated with stars of the eighth magnitude may be due to the fact that a considerable part of these proper motions have been taken from Boss's *P. G. C.*, and have been there included because a proper motion greater than the average has attracted attention to the stars and caused them to be observed; *i.e.* Boss's faint stars possess proper motions that are not typical. A similar influence may well be present in the AUWERS stars of the seventh magnitude, while it is certainly absent in the stars brighter than the fifth and fainter than the ninth magnitude. It should perhaps be here emphasized that the stars to be included in my observing programme were selected solely upon the basis of available early observations, usually those of W. STRUVE and O. STRUVE, who cannot be supposed to have been influenced in their choice by considerations of proper motions of the faint stars. There is therefore no initial bias here toward stars of large proper motion. In fact the rejection of the abnormally large motions, noted above, may have produced a slight uncompensated tendency in the opposite direction. Whatever view may be adopted with respect to minor variations in $m\mu$ it can hardly be doubted that over a wide range of magnitudes there is approximate constancy in this product. If the proper motions of the C stars are regarded as in the main fallacious, it seems remarkable that in point of magnitude they should conform to and numerically reproduce the law exhibited by proper motions whose genuineness has not been questioned. Assuming the relation $m\mu = C$ to be established and assuming further that, on the whole, faint and bright stars possess equal linear velocities, as seems to be indicated in the last column of Table VI, we are brought to the conclusion that these faint stars are much less remote than has hitherto been supposed; *i.e.* stellar distance is upon the average approximately proportional to stellar magnitude! This result seems so improbable that it may well be held under advisement until additional evidence is available.

I have further employed the relation $m\mu = C$ in examining the frequency law of distribution of proper motions among the approximately five hundred faint stars available. The frequency curve is, of necessity, ill determined at the ends, *i.e.* for very large and very small values of $m\mu$ but with due allowance for this uncertainty it may be inferred from the plotted curve, not here reproduced, that the value of $m\mu$ most frequently occuring (the mode) is not far from $15''$ and that approximately one-half of the stars show values of $m\mu$ greater than $35''$; *e. g.*, for stars of the tenth magnitude, the mode for proper motion is $1''.5$ per century, and approximately one-half of the tenth magnitude stars possess proper motions greater than $3''.5$ per century.

Eight years ago, I based upon the evidence furnished by only sixty-seven stars, the statement that the proper motions of the faint stars are of smaller amount in the galaxy than in extra-galactic regions, *Astron. Jour.*, No. 558. Boss subsequently announced a similar result from his study of the proper motions of the brighter stars, *Astron. Jour.*, No. 614, and SEELIGER has shown that this dependence of proper motion upon galactic latitude stands in close relationship with the relative numbers of stars in those regions, *Astr. Nachr.*, No. 4617. It is indeed self evident that if the finite stellar system is of greater extent in the galaxy than outside it, the average distance of galactic stars of a given magnitude should be greater and their average proper motion less, than the corresponding quantities for extra-galactic stars. Accepting such a relationship as real, I have applied it as a further test of the genuineness of the proper motions of the C stars here under consideration.

For each of SEELIGER's zones I–IX defined by the parallels of galactic latitude $+90° - +70° - +50° - +30° \;.\;.\;.\;.\; -70° - -90°$, I have found the mean value of $m\mu$ for all C stars within the zone, and these means together with the number of stars upon which they depend, n, are shown in the first two lines of Table X. The proper motions above designated as abnormally great are here included, and to eliminate any undue effect that they may exert, I have rejected from each zone all values of $m\mu$ greater than four times the mean value for the zone

in question and the zone similarly situated on the opposite side of the galaxy, repeating the criterion (never more than twice), until no further stars were rejected by it. The mean values of $m\mu$ obtained after this expurgation, are designated by the symbol C_1 in Table X and the number of stars included in each mean is designated n_1. As an indication of the magnitude of the larger proper motions, there is also given at the bottom of the table the values of $m\mu$ rejected as abnormally great.

Table X.

Relation of Proper Motion to the Galaxy.

Zone	I	II	III	IV	V	VI	VII	VIII	IX
Mean	97″	73″	58″	59″	36″	50″	47″	61″	109″
n	14	43	82	99	106	78	57	26	3
C_1	45	42	45	39	31	37	41	49	109
n_1	13	41	78	97	103	72	56	25	3
Rejecta	780″	908″	300″	220″	255″	190″	397″	371″	...
	...	512	563	1845	421	216	...	...	...
	...	...	189	...	193	367	...	...	...
	...	...	304	...	...	192	...	...	...
	...	...	...	...	...	207	...	...	...
	...	...	...	...	...	156	...	...	...

In view of the foregoing exhibit, there can be little doubt that the product $m\mu$, and the proper motions upon which it depends, show minimum values in the galaxy, Zone V, with an approximately symmetrical increase upon either side of it. I have therefore united into a single result the values of C_1 furnished by pairs of zones symmetrically placed with respect to the Milky Way and include these, under the rubric C_1, in Table XI.

In order to compare these results for the dependence of proper motion upon galactic latitude with those obtained by Boss from the stars of the *P. G. C.*, I have plotted the data given in *Astron. Jour.*, Vol. XXVI, p. 122, using as abscissæ the galactic latitude, b, that bisects each of Boss's zones, and as a corresponding ordinate, the product Mag. $\times$ μ_0 as there given. From the resulting curve, I have read the values of $m\mu$ shown in Table XI, under the rubric Boss. These quantities are not comparable with the numbers C_1 of that table since the two series involve and depend upon very different definitions of the abnormally great proper motion. Boss has employed a rather complicated criterion for the rejection of large proper motions, which is explained and illustrated at page 98 of the volume cited. Assuming that his eight illustrative numbers, there given, are fairly typical of his criterion, I have multiplied their mean value by the mean magnitude assigned to his stars ($16''.6 \times 5.7 = 94.62$), and obtain 95″ as the limiting value of $m\mu$ required in order to render C_1 comparable with Boss. Rejecting from the C stars all those that furnish values of $m\mu$ equal to, or greater than 95″, I obtain the mean values shown in Table XI, under the heading C_{11}. The agreement between these numbers and those of Boss is far too striking to be attributed to chance. Their substantial numerical agreement, although derived from stars of very different brightness, constitutes further confirmation of the relation $m\mu =$ Constant; while the slight excess of C_{11} over Boss may be regarded either as confirming Auwers' view that the second member of the preceding equation increases slowly with diminishing brightness, or as indicating that the adopted limit of rejection for abnormally large proper motions, 95″, should be somewhat reduced. The magnitudes of Boss being represented upon the average by larger numbers than those of the Harvard Scale would, of themselves, indicate that some reduction of this limit must be made for the C stars in order to render comparable the two series of results. But, numerically, the correction is not readily obtainable and I am constrained to omit it.

The degree to which proper motions are affected by galactic latitude is but little dependent upon the limit adopted for the rejection of abnormally great motions and it should be noted that C_1, C_{11} and Boss are in good agreement in showing that in the galaxy, the average proper motion is but little more than half as great as in high latitudes. Compare with this result Seeliger's conclusion, derived from enumeration of the *B.D.* stars: Dementsprechened würden auch die mittleren Eigenbewegungen, μ'', wenn sie, in Kilometern gemessen, uberall gleich sein oder nicht stark varireren würden, nur kaum die Hälfte betragen in der Milchstrasse, wie in den Polen. *loc. cit.*

Table XI.

Relation of Galactic Latitude to Proper Motion.

Zone	C_1	C_{11}	Boss
	″	″	″
I & IX	57	38	39
II VIII	44	38	38
III VII	43	37	36
IV VI	38	33	30
V	31	26	22

In form, the preceding discussion relates largely to the character that should be attributed to the observed motions of some five hundred faint stars. Are these motions in the main fictitious, reproducing only outstanding errors of observation and discussion; are they orbital in character, *e.g.*, due to the attraction of neighboring stars; or are they the undisturbed motions of independent bodies, such as are connoted by the term proper motion. It is not improbable that the first two alternatives above

stated, find expression among the motions of the *C* stars, but that they constitute any considerable portion of these motions seems wholly inconsistent with the evidence adduced above and recapitulated in the following summary. At least this evidence should place upon the skeptic, the burden of some proof for his case other than such generalities as have alone been used hitherto, *e.g.*, "When as usually happens, the measures of small stars in the field give a more or less different movement from that taken from the star catalogues, it would be manifestly absurd and in effect, begging the whole question, to ascribe the difference to the proper motion of the faint star." Is it not equally begging the question to assume that the faint stars are motionless? Proof is equally required for either alternative and the nature of the evidence here offered may be put into the words: If these proper motions were as absurd as is above suggested, any systematic discussion of them should lead to absurd and chaotic results. Such is not, however, the fact, and I am unable to reconcile with the foregoing discussions any doubt as to the substantial genuineness of these motions and their representative character with respect to stellar motion in general. Conceding them to be genuine and typical they lead, in part, to the following conclusions:

Summary.

1. Out of five hundred stars included between the seventh and thirteenth magnitudes, that have been observed for proper motion, approximately seventy-five per cent yield sensible proper motions.

2. These proper motions are referred to the system of Boss's *Preliminary General Catalogue* and possess a precision but little inferior to those of the fainter stars of that catalogue.

3. These proper motions confirm and extend from the lucid stars at least to the twelfth magnitude the relation, that in the mean, the amount of proper motion is inversely proportional to stellar magnitude. When seven per cent of the proper motions are rejected as abnormally great, the relation assumes the form $m\mu = 35''$, where μ is the mean centennial proper motion and the magnitudes, m, for the fainter stars conform to the Harvard Scale.

4. The frequency law of distribution of the product $m\mu$ for the faint stars here considered, is such that the value most frequently occurring is $15''$ and one-half of all values of the product are greater than $35''$. For both lucid and telescopic stars seven per cent of the values exceed $120''$.

5. The values of the proper motions show a marked dependence upon galactic latitude. The lucid and the telescopic stars agree in making the mean proper motion in high latitudes about twice as great as in the Milky Way.

6. The telescopic stars here discussed, furnish two determinations of the apex of solar motion, one from stars fainter than, the other from stars brighter than, the tenth magnitude. These determinations are in substantial agreement with the results furnished by brighter stars when account is taken of a progressive shift in the galactic longitude of the apex, indicated by the bright stars.

7. The Sun's linear velocity relative to the telescopic stars is substantially the same as its velocity relative to the lucid stars.

8. The stars between the seventh and thirteenth magnitudes share in the drift or preferential direction of motion found for the brighter stars and have approximately the same line of motion (vertex).

9. The numerical amount of preference for this direction of motion is substantially the same for bright and faint stars.

10. The lucid and telescopic stars have approximately equal components of motion perpendicular to the line of drift. They have also equal components of motion parallel to the line of drift.

11. The linear velocity of stellar motions is substantially independent of stellar magnitude.

12. The faint stars and bright stars are parts of one and the same stellar system and are in great measure intermingled, the faint stars being less remote than has been inferred from photometric considerations.

Washburn Observatory, September, 1913.

COMET *c* 1913 (*ZINNER*).

This comet has been identified with 1900 *c*.

Ephemeris.

G.M.T.	R.A.	Decl.
1913	h m s	° ′
Nov. 3.5	19 38 33	−15 08
7.5	20 02 49	−19 9

Elements.

T = 1913. Nov. 2.48 G.M.T.
$\omega = 171° 37'$
$\Omega = 191° 37$
$i = 33° 15'$
$q = 0.999$

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THE SECULAR PERTURBATIONS OF THE FOUR OUTER PLANETS,

By G. W. HILL.

The formulæ arrived at by LEVERRIER and STOCKWELL have never been employed in the construction of any planetary tables. This results from their large deviation from actuality. The question then arises whether they cannot be modified in such a way as to remove, in great measure, this defect. We wish to escape from the temporary makeshift of developments in powers of the time. GYLDÈN and POINCARÉ devoted themselves to the consideration of this matter; but the insistance on mathematical rigor leads to formulæ so entangled that they become unmanageable. Large concessions must therefore be made to feasability, although the result may be we shall be accused of empiricism.

Let φ be the angle of eccentricity, ω the orbit longitude of perihelion, i the inclination and θ the longitude of the ascending node. We adopt then the variables

$$h=2\sin\tfrac{1}{2}\varphi\sin\omega,\ l=2\sin\tfrac{1}{2}\varphi\cos\omega,\ p=2\sin\tfrac{1}{2}i\sin\theta\,,\quad q=2\sin\tfrac{1}{2}i\cos\theta.$$

The portion of the potential function Ω, which serves for the determination of secular perturbations arises only from the periodic developments of the reciprocals of the distances between the several planets considered, multiplied by the product of the masses at the end of the line. This very limited portion is yet, in all rigor, a very complicated function of the variables h, l, p, q. We, however, make the assumption that fairly useful formulæ for these variables may be obtained by supposing that Ω consists of two portions, one involving only the h and l, the other only the p and q; and that each portion is of a quadratic form in its variables. These assumptions are not justified unless we agree to modify the constants involved in the expressions in such a way as to bring about the closest agreement with actuality.

Let the two portions of Ω be Ω_1 and $-\Omega_2$, the first serving for the eccentricities and longitudes of the perihelia, the second for the inclination and longitudes of the nodes. The differential equations satisfied by the variables h, l, p, q are

$$\frac{dh}{dt}=c\frac{\partial\Omega_1}{\partial l},\quad \frac{dl}{dt}=-c\frac{\partial\Omega_1}{\partial h},\quad \frac{dq}{dt}=k\frac{\partial\Omega_2}{\partial p},\quad \frac{dp}{dt}=-k\frac{\partial\Omega_2}{\partial q},$$

The c and k are constants, but different for each planet. In the ordinary notation, they are expressed by the equations

$$c=\frac{na}{m\mu},\qquad k=\frac{na}{m\mu\sqrt{1-c^2}},$$

where, in the latter, to c is attributed its mean value.

Since Ω_1 is a symmetrical function of the h and l, and Ω_2 of p and q, it is necessary to write in Ω_1 only the terms involving the l, and in Ω_2 only the terms involving the p. Thus we may put

$$\begin{aligned}\Omega_1=\tfrac{1}{2}[0,0]l^2+\tfrac{1}{2}[1,1]l'^2&+\tfrac{1}{2}[2,2]l''^2+\tfrac{1}{2}[3,3]l'''^2\\ -[0,1]ll'&-[0,2]ll''-[0,3]ll'''\\ &-[1,2]l'l''-[1,3]l'l'''\\ &-[2,3]l''l'''\end{aligned}$$

$$\begin{aligned}\Omega_2=\tfrac{1}{2}[0,0]p^2+\tfrac{1}{2}[1,1]p'^2&+\tfrac{1}{2}[2,2]p''^2+\tfrac{1}{2}[3,3]p'''^2\\ -[0,1]pp'&-[0,2]pp''-[0,3]pp'''\\ &-[1,2]p'p''-[1,3]p'p'''\\ &-[2,3]p''p'''\end{aligned}$$

in which it is to be noted that the constants denoted by the integers in brackets are not necessarily identical in Ω_1 and Ω_2. The accents attached to the symbols are distributed in the order of *Jupiter*, *Saturn*, *Uranus* and *Neptune*. The constants denoted by the brackets are all positive.

Thus we can write the differential equations

$$\frac{dh}{dt}=c\ [0,0]l-c\ [0,1]l'-c\ [0,2]l''-c\ [0,3]l'''$$

$$\frac{dh'}{dt}=-c'\ [0,1]l+c'\ [1,1]l'-c'\ [1,2]l''-c'\ [1,3]l'''$$

$$\frac{dh''}{dt} = -c''[0,2]l - c''[1,2]l' + c''[2,2]l'' - c''[2,3]l'''$$

$$\frac{dh'''}{dt} = -c'''[0,3]l - c'''[1,3]l' - c'''[2,3]l'' + c'''[3,3]l'''$$

The four equations giving the motions of the l are obtained from these by interchanging the h and l and reversing the sign of the right members.

For the movement of the q we have the similar equations

$$\frac{dq}{dt} = k[0,0]p - k[0,1]p' - k[0,2]p'' - k[0,3]p'''$$

$$\frac{dq'}{dt} = -k'[0,1]p + k'[1,1]p' - k'[1,2]p'' - k'[1,3]p'''$$

$$\frac{dq''}{dt} = -k''[0,2]p - k''[1,2]p' + k''[2,2]p'' - k''[2,3]p'''$$

$$\frac{dq'''}{dt} = -k'''[0,3]p - k'''[1,3]p' - k'''[2,3]p'' + k'''[3,3]p'''$$

The equations for the p are obtained by interchanging the p and q and reversing the sign of the right members.

It is convenient to merge the constants c and k in the constants denoted by the integers in bracket; we shall employ this abbreviation and write the preceding equation thus:

$$\frac{dh}{dt} = [0,0]l - [0,1]l' - [0,2]l'' - [0,3]l'''$$

$$\frac{dh'}{dt} = -[1,0]l + [1,1]l' - [1,2]l'' - [1,3]l'''$$

$$\frac{dh''}{dt} = -[2,0]l - [2,1]l' + [2,2]l'' - [2,3]l'''$$

$$\frac{dh'''}{dt} = -[3,0]l - [3,1]l' - [3,2]l'' + [3,3]l'''$$

$$\frac{dq}{dt} = [0,0]p - [0,1]p' - [0,2]p'' - [0,3]p'''$$

$$\frac{dq'}{dt} = -[1,0]p + [1,1]p' - [1,2]p'' - [1,3]p'''$$

$$\frac{dq''}{dt} = -[2,0]p - [2,1]p' + [2,2]p'' - [2,3]p'''$$

$$\frac{dq'''}{dt} = -[3,0]p - [3,1]p' - [3,2]p'' + [3,3]p'''$$

in which it must be borne in mind that by inverting the integers between the brackets we obtain a different constant.

We must now assemble the data on which the following investigation is founded. Adopting, as the epoch, the mean noon of December 31, 1899, at Greenwich, we have the table of elements of the four planets

	Recip. of *Mass*	Sid. Mean Mot.		Mean Long. at Epoch.	Eccentricity	Long. Per.	Inclination	Long. Node
	$\frac{m}{m}$	n	Com. Log. a	L	e	ω	i	θ
Jupiter	1047.355	109256″.6152	0.71623326	238° 2′59″.41	0.04833560	12°42′43″.85	1°18′31″.89	99°26′32″.87
Saturn	3501.6	43996 .0875	0.97967814	266 33 56 .93	0.05588468	91 5 26 .88	2 29 32 .59	112 47 10 .82
Uranus	22869	15424 .8680	1.28312009	244 12 43 .46	0.04632812	171 34 39 .9	0 46 20 .62	73 28 58 .3
Neptune	19700	7865 .6571	1.47811178	84 29 31 .72	0.00896745	46 42 19 .6	1 46 44 .84	130 39 23 .3

The elements of *Uranus* and *Neptune*, exclusive of the masses, are derived from the tables of LEVERRIER. In NEWCOMB's tables of these planets the great inequality of period 4,000 years is merged in the secular perturbations. It being difficult to estimate the corrections to be applied, I have had recourse to LEVERRIER. On this account a difference will be noticed in the n and a of *Uranus* and *Neptune* here given and those stated in a previous communication (*A.J.*, No. 646). The elements of *Jupiter* and *Saturn* are derived from my own investigation.

In accordance with a previous suggestion it has been decided to refer the planes of the orbits to the fixed invariable plane of the four planets, whose position with respect to the ecliptic of the beginning of 1900 is given by the data

$$\gamma = 1°\ 35'\ 1''.258\,, \qquad \Pi = 106°\ 37'\ 50''.48$$

	Cor. to Orbit Longs.	Log (2 sin ½ϕ)	ω	Log (2 sin ½i)	i	θ
Jupiter	+ 8″.15	8.6843941	12°42′52″.00	7.7588322	19′43″.75	316°30′37″.87
Saturn	−13 .29	8.7471625	91 5 13 .59	8.2119339	56 0 .19	123 15 41 .14
Uranus	+20 .95	8.6659612	171 35 0 .85	8.2537598	61 39 .91	310 53 43 .08
Neptune	−36 .07	7.9526736	46 11 43 .53	8.1024819	43 31 .62	193 21 37 .15

In order to have a departure point for the measurement of orbit longitudes the mean equinox of the epoch will be turned down (*rabatu*) from the ecliptic onto the invariable plane. In this way the orbit longitudes do not undergo a large change. Its amount is stated in the foregoing table of elements referred to the invariable plane

For the sake of comparison and the desired adjustment of constants it is necessary to know the results of the application of MACLAURIN's theorem to the integration of the differential equations by series arranged according to ascending powers of the time. In the cases of *Jupiter* and *Saturn* my own investigations have led to the formulæ for the elements in the planes of the orbits:

$$\begin{aligned}
e &= 0.04825336 + 0.0001646948\,T - 0.0000004500T^2 + 0.00000000284T^3 \\
\omega &= 11°\,54'\,26''.67 + 766''.1793\,T + 2''.63845T^2 - 0''.002894T^3 \\
e' &= 0.05605744 - 0.0003452494\,T - 0.0000005631\,T^2 - 0.00000000687\,T^3 \\
\omega' &= 90°\,6'\,38''.45 + 2029''.676\,T + 2''.05127\,T^2 + 0''.069357\,T^3 .
\end{aligned}$$

Here T is in units of a 100 Julian years and is counted from the beginning of 1850. When T is counted from 1900 instead of 1850 and 50 years of precession added to the longitudes, these formulæ are changed to

$$\begin{aligned}
e &= 0.04833560 + 0.0001642469\,T - 0.0000004457\,T^2 + 0.00000000284\,T^3 \\
\omega &= 12°\,42'\,43''.85 + 768''.8156\,T + 2''.63411\,T^2 - 0''.002894\,T^3 \\
e' &= 0.05588468 - 0.0003458177\,T - 0.0000005734\,T^2 - 0.00000000687\,T^3 \\
\omega' &= 91°\,5'\,26''.88 + 2031''.779\,T + 2''.15531\,T^2 + 0''.069357\,T^3
\end{aligned}$$

They afford the following values for the variables at the assigned dates; but the previously mentioned correction to orbits longitudes has been applied to ω and ω':

Date	e	ω	e'	ω'
900	0.04664572	10°39′10″.15	0.05929239	85°29′ 1″.97
1400	4750287	11 39 54 .14	5760029	88 16 39 .91
1900	4833560	12 42 52 .00	5588468	91 5 13 .59
2400	4914605	13 48 1 .57	5414040	93 55 35 .04
2900	4993634	14 55 20 .67	5236229	96 48 36 .27

The position of the planes of the orbits of *Jupiter* and *Saturn* with respect to the ecliptic and equinox of 1900 are given by the formulæ:

$$\begin{aligned}
\sin i\ \sin\theta &= \sin i_0 \sin\theta_0 - 9''.5040T + 0''.08616T^2 + 0''.000244T^3 \\
\sin i\ \cos\theta &= \sin i_0 \cos\theta_0 - 13\ .0043T - 0\ .06673T^2 + 0\ .000321T^3 \\
\sin i'\sin\theta' &= \sin i_0'\sin\theta_0' + 24\ .1087T - 0\ .21351T^2 - 0\ .000652T^3 \\
\sin i'\cos\theta' &= \sin i_0'\cos\theta_0' + 33\ .2538T + 0\ .16374T^2 - 0\ .000851T^3
\end{aligned}$$

where the sines of the inclinations are reckoned in sexagesimal seconds. The formulæ lead to the following values at the assigned dates:

	i	θ	i'	θ'
900	1° 19′ 55″.733	97° 47′ 21″.18	2° 27′ 41″.915	115° 19′ 22″.02
1400	1 19 11 .012	98 35 27 .40	2 28 41 .527	114 3 33 .68
1900	1 18 31 .892	99 26 32 .87	2 29 32 .588	112 47 10 .82
2400	1 17 58 .667	100 20 19 .71	2 30 15 .081	111 30 17 .40
2900	1 17 34 .619	101 16 27 .64	2 30 48 .950	110 12 56 .03

These values were reduced from the ecliptic and mean equinox of 1900 to the invariable plane, but by employing the data $\gamma = 1°\,35'\,1''.266$, $\Pi = 106°\,57'\,56''.75$, which resulted from using NEWCOMB's elements of *Uranus* and *Neptune* instead of those previously mentioned. The difference is of no moment. The results are

	i		θ		i'		θ'	
900	20′ 12″.388		324° 4′ 51″.63		55′ 39″.064		130° 16′ 0″.87	
		−14″.241		−3°45′40″.56		+11″.029		−3°30′18″.79
1400	19 58 .147		320 19 11 .07		55 50 .093		126 45 42 .08	
		14 .397		3 48 23 .30		10 .087		3 30 0 .94
1900	19 43 .750		316 30 37 .87		56 0 .190		123 15 41 .14	
		14 .494		3 51 45 .44		9 .185		3 29 49 .29
2400	19 29 .256		312 38 52 .43		56 9 .375		119 45 51 .85	
		14 .499		3 55 17 .69		8 .289		3 29 45 .97
2900	19 14 757		308 43 34 .74		56 17 .664		116 16 5 .88	

The secular perturbations of *Uranus* and *Neptune* were derived from LEVERRIER's discussion (*Annales de l'Observatoire de Paris, Tom.* XI) by adapting his formulæ to our values of the planetary masses. Counting T from 1850, the expressions for the elements in the planes of the orbits are

$$e'' = e_0'' - 5''.5104\,T + 0''.01576\,T^2 \qquad \omega'' = \omega_0'' + 309''.269\,T - 0''.3184\,T^2$$
$$e''' = e_0''' + 1\;.2455\,T + 0\;.00015\,T^2 \qquad \omega''' = \omega_0''' + 85\;.169\,T + 0\;.2678\,T^2$$

Reduced to the epoch 1900, they become

$$e'' = 9555''.86 - 5''.49464\,T + 0''.01576\,T^2 \qquad \omega'' = 171^\circ\,34'\,39''.9 + 308''.9506\,T - 0''.3184\,T^2$$
$$e''' = 1849\;.67 + 1\;.24665\,T + 0\;.00015\,T^2 \qquad \omega''' = 46\;42\;19\;.6 + 85\;.4368\,T + 0\;.2678\,T^2$$

Referred to the ecliptic and equinox of 1850, the elements defining the position of the planes of the orbits are

$$i'' = i_0'' - 5''.9744\,T + 0''.01280\,T^2 \qquad \theta'' = \theta_0'' + 272''.285\,T + 1''.4860\,T^2$$
$$i''' = i_0''' + 0\;.7967\,T + 0\;.00095\,T^2 \qquad \theta''' = \theta_0''' - 21\;.728\,T - 0\;.0079\,T^2$$

Reduced to the ecliptic and equinox of 1900 and T being counted from the same date, they become

$$i'' = 0^\circ\,46'\,20''.62 - 5''.9923\,T + 0''.01263\,T^2 \qquad \theta'' = 73^\circ\,28'\,58''.3 + 269''.762\,T + 1''.4843\,T^2$$
$$i''' = 1\;46\;44\;.84 + 0\;.7994\,T + 0\;.00099\,T^2 \qquad \theta''' = 130\;39\;23\;.3 - 21\;.729\,T - 0\;.0079\,T^2$$

These formulæ produce the following values for the specified dates:

	i''	θ''	i'''	θ'''
900	47′21″.806	72°46′29″.11	1°46′36″.945	130°42′59″.80
1400	46 50 .897	73 7 6 .60	40 .868	41 11 .75
1900	46 20 .620	73 28 58 .30	40 .840	39 23 .30
2400	45 50 .975	73 53 4 .22	48 .862	37 34 .45
2900	45 21 .960	74 16 24 .35	52 .933	35 45 .32

Changed to the invariable plane with the elements $\gamma = 1^\circ\,35'\,1''.258$, $\Pi = 106^\circ\,37'\,50''.48$, they give

	i''		θ''		i'''		θ'''	
900	61′37″.321		311°59′ 4″.55		43′33″.984		193°34′54″.08	
		+1″.531		−32′44″.65		−1″.190		−6′36″.35
1400	38 .852		311 26 19 .90		32 .794		28 17 .73	
		1 .055		32 36 .82		1 .174		6 40 .58
1900	39 .907		310 53 43 .08		31 .620		21 37 .15	
		0 .584		32 30 .16		1 .157		6 44 .82
2400	40 .491		310 21 12 .92		30 .463		14 52 .33	
		0 .170		32 24 .66		1 .140		6 49 .03
2900	40 .661		309 48 48 .26		29 .323		8 3 .30	

We now proceed to the adjustment of the coefficients denoted by integers enclosed in brackets. The formulæ used are (putting E for 2 sin $\frac{1}{2}\varphi$)

$$\frac{d\text{E}}{dt} = [0\,,1]\,\text{E}'\sin(\omega'-\omega) + [0\,,2]\,\text{E}''\sin(\omega''-\omega) + [0\,,3]\,\text{E}'''\sin(\omega'''-\omega)$$

$$\frac{d\text{E}'}{dt} = [1\,,0]\,\text{E}\sin(\omega-\omega') + [1\,,2]\,\text{E}''\sin(\omega''-\omega') + [1\,,3]\,\text{E}'''\sin(\omega'''-\omega')$$

$$\frac{d\text{E}''}{dt} = [2\,,0]\,\text{E}\sin(\omega-\omega'') + [2\,,1]\,\text{E}'\sin(\omega'-\omega'') + [2\,,3]\,\text{E}'''\sin(\omega'''-\omega'')$$

$$\frac{d\text{E}'''}{dt} = [3\,,0]\,\text{E}\sin(\omega-\omega''') + [3\,,1]\,\text{E}'\sin(\omega'-\omega''') + [3\,,2]\,\text{E}''\sin(\omega''-\omega''')$$

$$\frac{d\omega}{dt} = [0\,,0] - [0\,,1]\frac{\text{E}'}{\text{E}}\cos(\omega'-\omega) - [0\,,2]\frac{\text{E}''}{\text{E}}\cos(\omega''-\omega) - [0\,,3]\frac{\text{E}'''}{\text{E}}\cos(\omega'''-\omega)$$

$$\frac{d\omega'}{dt} = [1\,,1] - [1\,,0]\frac{\text{E}}{\text{E}'}\cos(\omega-\omega') - [1\,,2]\frac{\text{E}''}{\text{E}'}\cos(\omega''-\omega') - [1\,,3]\frac{\text{E}'''}{\text{E}'}\cos(\omega'''-\omega')$$

$$\frac{d\omega''}{dt} = [2\,,2] - [2\,,0]\frac{\text{E}}{\text{E}''}\cos(\omega-\omega'') - [2\,,1]\frac{\text{E}'}{\text{E}''}\cos(\omega'-\omega'') - [2\,,3]\frac{\text{E}'''}{\text{E}''}\cos(\omega'''-\omega'')$$

$$\frac{d\omega'''}{dt} = [3\,,3] - [3\,,0]\frac{\text{E}}{\text{E}'''}\cos(\omega-\omega''') - [3\,,1]\frac{\text{E}'}{\text{E}'''}\cos(\omega'-\omega''') - [3\,,2]\frac{\text{E}''}{\text{E}'''}\cos(\omega''-\omega''')$$

and I for 2 sin $\frac{1}{2}I'$

$$\frac{d\text{I}}{dt} = -[0\,,1]\,\text{I}'\sin(\theta'-\theta) - [0\,,2]\,\text{I}''\sin(\theta''-\theta) - [0\,,3]\,\text{I}'''\sin(\theta'''-\theta)$$

$$\frac{d\text{I}'}{dt} = -[1\,,0]\,\text{I}\sin(\theta-\theta') - [1\,,2]\,\text{I}''\sin(\theta''-\theta') - [1\,,3]\,\text{I}'''\sin(\theta'''-\theta')$$

$$\frac{d\text{I}''}{dt} = -[2\,,0]\,\text{I}\sin(\theta-\theta'') - [2\,,1]\,\text{I}'\sin(\theta'-\theta'') - [2\,,3]\,\text{I}'''\sin(\theta'''-\theta'')$$

$$\frac{d\text{I}'''}{dt} = -[3\,,0]\,\text{I}\sin(\theta-\theta''') - [3\,,1]\,\text{I}'\sin(\theta'-\theta''') - [3\,,2]\,\text{I}''\sin(\theta''-\theta''')$$

$$\frac{d\theta}{dt} = -[0\,,0] + [0\,,1]\frac{\text{I}'}{\text{I}}\cos(\theta'-\theta) + [0\,,2]\frac{\text{I}''}{\text{I}}\cos(\theta''-\theta) + [0\,,3]\frac{\text{I}'''}{\text{I}}\cos(\theta'''-\theta)$$

$$\frac{d\theta'}{dt} = -[1\,,1] + [1\,,0]\frac{\text{I}}{\text{I}'}\cos(\theta-\theta') + [1\,,2]\frac{\text{I}''}{\text{I}'}\cos(\theta''-\theta') + [1\,,3]\frac{\text{I}'''}{\text{I}'}\cos(\theta'''-\theta')$$

$$\frac{d\theta''}{dt} = -[2\,,2] + [2\,,0]\frac{\text{I}}{\text{I}''}\cos(\theta-\theta'') + [2\,,1]\frac{\text{I}'}{\text{I}''}\cos(\theta'-\theta'') + [2\,,3]\frac{\text{I}'''}{\text{I}''}\cos(\theta'''-\theta'')$$

$$\frac{d\theta'''}{dt} = -[3\,,3] + [3\,,0]\frac{\text{I}}{\text{I}'''}\cos(\theta-\theta''') + [3\,,1]\frac{\text{I}'}{\text{I}'''}\cos(\theta'-\theta''') + [3\,,2]\frac{\text{I}''}{\text{I}'''}\cos(\theta''-\theta''')$$

The variables in all these equations, as well as their differentials in the left members, must receive the values they have at the epoch 1900.

Since the action of *Uranus* and *Neptune* on *Jupiter* and *Saturn* is only about one per cent. of the action of these planets on each other, in favor of having the simplest possible expressions for the elements of the latter planets, we shall assume that the following coefficients vanish, viz., that in both sets of equations $[0\,,2] = [0\,,3] = [1\,,2] = [1\,,3] = 0$. This assumption approximately merges the action of *Uranus* and *Neptune* in the mutual action of *Jupiter* and *Saturn*. As the number of equations is insufficient to determine corrections to all the constants we assume that [2 , 0] , [2 , 3] , [3 , 0] and [3 , 1] need no correction in either case. It is convenient to express the coefficients in sexagesimal seconds, as has been done by previous investigators. The coefficients of the linear equations are denoted by their common logarithms enclosed in brackets and 10 has been added to avoid negative characteristics. Considering first the elements in the planes of the orbits, the equations of adjustment are

$$
\begin{aligned}
[9.5303033] &= [8.7384577]\ [0\,,1] \\
7''.688156 &= [0\,,0] - [9.3674410]\ [0\,,1] \\
-[9.8537818] &= -\ [8.6753893]\ [1\,,0] \\
20''.317790 &= [1\,,1] - [9.2113041]\ [1\,,0] \\
-\ 0.0549907 &= -\ [8.2412986]\ [2\,,0] - [8.7414608]\ [2\,,1] - [7.8666306]\ [2\,,3] \\
3.089506 &= [2\,,2] + [9.9882025]\ [2\,,0] - [9.2992709]\ [2\,,1] + [9.0440904]\ [2\,,3] \\
0''.01246688 &= -\ [8.4317418]\ [3\,,0] + [8.5922873]\ [3\,,1] + [8.5799182]\ [3\,,2] \\
0\ .854368 &= [3\,,3] - [0.6503919]\ [3\,,0] - [0.6488364]\ [3\,,1] - [0.4706656]\ [3\,,2]
\end{aligned}
$$

The values of the constants were computed by the usual formulæ such as LEVERRIER and STOCKWELL used, with the exception that in the second set the factor $(1 - e^2)^{-4}$ was taken into account. These values for [2 , 0] , [2 , 3] , [3 , 0] , [3 , 1] were substituted in the preceding equations, with the result that we have eight equations for the discovery of as many unknowns. The solution leads to the following exhibit, where the unadjusted values are added by way of comparison:

	Unadjusted	Adjusted		Unadjusted	Adjusted
log [0 , 0]	0.8739058	0.9605290	log [2 , 0]	9.4969786	9.4969786
log [0 , 1]	0.6830050	0.7918456	log [2 , 1]	9.9227509	9.9329198
log [0 , 2]	8.4411727	0	log [2 , 2]	0.4377243	0.4654383
log [0 , 3]	7.9611608	0	log [2 , 3]	9.4892608	9.4892608
log [1 , 0]	1.0753215	1.1783926	log [3 , 0]	8.5844895	8.5844895
log [1 , 1]	1.2687665	1.3607112	log [3 , 1]	8.9084825	8.9084825
log [1 , 2]	9.2594615	0	log [3 , 2]	9.3269836	9.4344793
log [1 , 3]	8.4074703	0	log [3 , 3]	9.8246875	9.7657884

Treating, in the second place, the elements defining the planes of the orbits, the equations of adjustment are

$$
\begin{aligned}
-[8.4611364] &= -\ [7.5721201]\ [0\,,1] \\
-\ 27''.612040 &= -\ [0\,,0] - [0.4413854]\ [0\,,1] \\
[8.2850025] &= [7.1190184]\ [1\,,0] \\
-\ 25''.187810 &= -\ [1\,,1] - [9.5351820]\ [1\,,0] \\
[7.2149825] &= -\ [6.7493809]\ [2\,,0] - [7.3352705]\ [2\,,1] + [8.0502729]\ [2\,,3] \\
-\ 3''.914209 &= -\ [2\,,2] + [9.5029834]\ [1\,,0] - [9.9543079]\ [2\,,1] - [9.5136359]\ [2\,,3] \\
-[7.36743] &= -\ [7.6816823]\ [3\,,0] + [8.1851919]\ [3\,,1] - [8.2015508]\ [3\,,2] \\
-\ 0''.80540 &= -\ [3\,,3] - [9.3942067]\ [3\,,0] + [9.6414377]\ [3\,,1] - [9.8161917]\ [3\,,2]
\end{aligned}
$$

Following the same course as in the preceding group of equations we have the values

	Unadjusted	Adjusted		Unadjusted	Adjusted
log [0 , 0]	0.8743647	0.7932759	log [2 , 0]	9.9716900	9.9716900
log [0 , 1]	0.8682192	0.8890163	log [2 , 1]	0.1417394	0.0542872
log [0 , 2]	8.9159119	0	log [2 , 2]	0.4383554	0.4854788
log [0 , 3]	8.3582320	0	log [2 , 3]	9.6142587	9.6142587
log [1 , 0]	1.2608391	1.1659841	log [3 , 0]	9.2511242	9.2511242
log [1 , 1]	1.2695288	1.3045124	log [3 , 1]	9.3157130	9.3157130
log [1 , 2]	9.4815812	0	log [3 , 2]	9.4513728	9.4652105
log [1 , 3]	8.8154407	0	log [3 , 3]	9.8247094	9.8199727

Attending now to the integration of our two systems of linear differential equations, we take up that which is concerned with the eccentricities and longitudes of perihelia. The equation of the fourth degree in g, which furnishes the rates of motion of the four arguments in the expression of the variables, separates into two quadratics on account of the vanishing of the four coefficients $[0\,,2]$, $[0\,,3]$, $[1\,,2]$, $[1\,,3]$. These quadratics are, first

$$\{g - [0\,,0]\}\,\{g - [1\,,1]\} - [0\,,1]\,[1\,,0] = 0$$

which is more especially concerned with the interaction of *Jupiter* and *Saturn*. Its roots are

$$g = 27''.916861\,, \qquad g_1 = 4''.160584.$$

The second quadratic is

$$\{g - [2\,,2]\}\,\{g - [3\,,3]\} - [2\,,3]\,[3\,,2] = 0.$$

Its roots are

$$g_2 = 2''.955734\,, \qquad g_3 = 0''.547800$$

Supposing that a particular solution of the system of differential equations is represented by

$$h = M\sin(gt+\beta)\,,\quad l = M\cos(gt+\beta)\,,\quad h' = M'\sin(gt+\beta)\,,\quad l' = M'\cos(gt+\beta)\,,\quad \text{etc.,}$$

where M, M', . . β are constants, it is obvious that the M and g satisfy the conditions

$$\begin{aligned}
&\{g-[0{,}0]\}\,M + [0\,,1]\,M' &&= 0\\
&[1\,,0]\,M + \{g-[1\,,1]\}\,M' &&= 0\\
&[2\,,0]\,M + [2\,,1]\,M' + \{g-[2\,,2]\}\,M'' + [2\,,3]\,M''' &&= 0\\
&[3\,,0]\,M + [3\,,1]\,M' + [3\,,2]\,M'' + \{g-[3\,,3]\}\,M''' &&= 0
\end{aligned}$$

By attaching the subscripts 1, 2, 3 in succession to g, the M and the β, we get three more groups of four equations each, which must be satisfied by the g and the M having severally these subscripts. These sixteen equations, for the present case are

$$\begin{aligned}
&[1.2738259]\,M + [0.7918456]\,M' &&= 0\\
&[1.1783926]\,M + [0.6964122]\,M' &&= 0\\
&[9.4969786]\,M + [9.9329198]\,M' + [1.3978790]\,M'' + [9.4892608]\,M''' &&= 0\\
&[8.5844895]\,M + [8.9084825]\,M' + [9.4344793]\,M'' + [1.4366984]\,M''' &&= 0\\
-&[0.6964122]\,M_1 + [0.7918456]\,M_1' &&= 0\\
&[1.1783926]\,M_1 - [1.2738259]\,M_1' &&= 0\\
&[9.4969786]\,M_1 + [9.9329198]\,M_1' + [0.0934955]\,M_1'' + [9.4892608]\,M_1''' &&= 0\\
&[8.5844895]\,M_1 + [8.9084825]\,M_1' + [9.4344793]\,M_1'' + [0.5535703]\,M_1''' &&= 0\\
-&[0.7906713]\,M_2 + [0.7918456]\,M_2' &&= 0\\
&[1.1783926]\,M_2 - [1.3008234]\,M_2' &&= 0\\
&[9.4969786]\,M_2 + [9.9329198]\,M_2' + [8.5485206]\,M_2'' + [9.4892608]\,M_2''' &&= 0\\
&[8.5844895]\,M_2 + [8.9084825]\,M_2' + [9.4344793]\,M_2'' + [0.3752195]\,M_2''' &&= 0\\
-&[0.9475785]\,M_3 + [0.7918456]\,M_3' &&= 0\\
&[1.1783926]\,M_3 - [1.3555690]\,M_3' &&= 0\\
&[9.4969786]\,M_3 + [9.9329198]\,M_3' - [0.3752195]\,M_3'' + [9.4892608]\,M_3''' &&= 0\\
&[8.5844895]\,M_3 + [8.9084825]\,M_3' + [9.4344793]\,M_3'' - [8.5485206]\,M_3''' &&= 0
\end{aligned}$$

This system of equations cannot be satisfied unless $M_2 = M_2' = M_3 = M_3' = 0$. Moreover, the first and second are identical, as also are the fifth and sixth, the ninth and tenth, the fifteenth and sixteenth. The eight remaining independent equations can be put in the form

$$\begin{aligned}
M' &= -[0.4819803]\,M & M_1' &= [9.9045666]\,M_1 & M_2''' &= -[9.0592598]\,M_2''\\
M'' &= [8.9607156]\,M & M_1'' &= -[9.9117245]\,M_1 & M_3''' &= [0.8859587]\,M_3''\\
M''' &= [7.8245075]\,M & M_1''' &= [8.5201094]\,M_1
\end{aligned}$$

These eight equations with the eight furnished by the initial conditions determine the twelve M and the four β. The latter are, by making $t = 0$ in the general formulæ:

$$
\begin{array}{rlllll}
 & M\sin\beta + & M_1\sin\beta_1 & & & = [8.0269986] \\
-[0.4819803] & M\sin\beta + [9.9045666] & M_1\sin\beta_1 & & & = [8.7473843] \\
 & M\cos\beta + & M_1\cos\beta_1 & & & = [8.6736120] \\
-[0.4819803] & M\cos\beta + [9.9045666] & M_1\cos\beta_1 & & & = -[7.0255867] \\
[8.9607156] & M\sin\beta - [9.9117245] & M_1\sin\beta_1 + & M_2''\sin\beta_2 + & M_3''\sin\beta_3 & = [7.8314035] \\
[8.9607156] & M\cos\beta - [9.9117245] & M_1\cos\beta_1 + & M_2''\cos\beta_2 + & M_3''\cos\beta_3 & = -[8.6612587] \\
[7.8245075] & M\sin\beta + [8.5201094] & M_1\sin\beta_1 - [9.0592598] & M_2''\sin\beta_2 + [0.8859587] & M_3''\sin\beta_3 & = [7.8146367] \\
[7.8245075] & M\cos\beta + [8.5201094] & M_1\cos\beta_1 - [9.0592598] & M_2''\cos\beta_2 + [0.8859587] & M_3''\cos\beta_3 & = [7.7889196]
\end{array}
$$

From these equations are derived the following:

$$
\begin{array}{llll}
M\sin\beta = -[8.0914279] & M_1\sin\beta_1 = [8.3614369] & M_2''\sin\beta_2 = [8.4069908] & M_3''\sin\beta_3 = [7.0571792] \\
M\cos\beta = [8.0062458] & M_1\cos\beta_1 = [8.5684282] & M_2''\cos\beta_2 = -[8.2288412] & M_3''\cos\beta_3 = [6.5787442]
\end{array}
$$

whence

$$
\begin{array}{llll}
M = [8.2035022] & M_1 = [8.6392305] & M_2'' = [8.4862100] & M_3'' = [7.0799269] \\
\beta = 309^\circ\, 25'\, 0''.19 & \beta_1 = 31^\circ\, 50'\, 7''.37 & \beta_2 = 123^\circ\, 33'\, 53''.47 & \beta_3 = 71^\circ\, 37'\, 1''.14
\end{array}
$$

Then if, for brevity, we adopt the notation

$$
\begin{array}{ll}
\chi = 309^\circ\, 25'\;\; 0''.19 + 27''.916861t\,, & \chi_1 = 31^\circ\, 50'\, 7''.37 + 4''.160584t\,, \\
\chi_2 = 123\;\; 33\;\; 53\;\; .47 + \;\;2\;\; .955734t\,, & \chi_3 = 71\;\; 37\;\; 1\;\; .14 + 0\;\; .547800t\,,
\end{array}
$$

we shall have

$$
\begin{aligned}
\text{E}\;\;\sin\omega &= 0.01597726\sin\chi + 0.04357431\sin\chi_1 \\
\text{E}\;\;\cos\omega &= 0.01597726\cos\chi + 0.04357431\cos\chi_1 \\
\text{E}'\;\sin\omega' &= -0.04847106\sin\chi + 0.03497817\sin\chi_1 \\
\text{E}'\;\cos\omega' &= -0.04847106\cos\chi + 0.03497817\cos\chi_1 \\
\text{E}''\sin\omega'' &= 0.00145955\sin\chi - 0.03555945\sin\chi_1 + 0.03063444\sin\chi_2 + 0.00120206\sin\chi_3 \\
\text{E}''\cos\omega'' &= 0.00145955\cos\chi - 0.03555945\cos\chi_1 + 0.03063444\cos\chi_2 + 0.00120206\cos\chi_3 \\
\text{E}'''\sin\omega''' &= 0.00010666\sin\chi + 0.00144324\sin\chi_1 - 0.00351132\sin\chi_2 + 0.00924455\sin\chi_3 \\
\text{E}'''\cos\omega''' &= 0.00010666\cos\chi + 0.00144324\cos\chi_1 - 0.00351132\cos\chi_2 + 0.00924455\cos\chi_3
\end{aligned}
$$

The treatment of the equations concerned with the inclinations and longitudes of nodes is quite similar to the foregoing; the only difference worth mention is that here the four g's have negative values. As before, the biquadratic in g separates into two quadratics, of which the first

$$\{g + [0\,,0]\}\,\{g + [1\,.\,1]\} - [0\,,1]\,[1\,,0] = 0$$

is more especially concerned with the interaction of *Jupiter* and *Saturn*. Its roots are

$$g = -\,25''.921343\,, \qquad g_1 = -0''.453700.$$

The second is

$$\{g + [2\,,2]\}\,\{g + [3\,,3]\} - [2\,,3]\,[3\,,2] = 0$$

Its roots are

$$g_2 = -\,3''.107369\,, \qquad g_3 = -\,0''.611574.$$

Making a like substitution in the differential equations as before, we omit to write the equations which only establish that M_2, M_2', M_3, M_3' vanish. The twelve remaining are

$$
\begin{array}{llllll}
[1.2946582]\,M & + [0.8890163]\,M' & & & = 0 \\
[1.1659841]\,M & + [0.7603422]\,M' & & & = 0 \\
[9.9716900]\,M & + [0.0502872]\,M' & + [1.3591342]\,M'' & + [9.6142587]\,M''' & = 0 \\
[9.2511242]\,M & + [9.3157130]\,M' & + [9.4652105]\,M'' & + [1.4024452]\,M''' & = 0
\end{array}
$$

$$
\begin{array}{llllr}
-[0.7603422]\,M_1 + [0.8890163]\,M_1' & & & & = 0 \\
[1.1659841]\,M_1 - [1.2946582]\,M_1' & & & & = 0 \\
[9.9716900]\,M_1 + [0.0542872]\,M_1' & - [0.4157395]\,M_1'' & + [9.6142587]\,M_1''' & & = 0 \\
[9.2511242]\,M_1 + [9.3157130]\,M_1' & + [9.4652105]\,M_1'' & - [9.3158696]\,M_1''' & & = 0 \\
 & [8.6908855]\,M_2'' & + [9.6142587]\,M_2''' & & = 0 \\
 & [9.4652105]\,M_2'' & + [0.3885837]\,M_2''' & & = 0 \\
 & -[0.3885837]\,M_3'' & + [9.6142587]\,M_3''' & & = 0 \\
 & [9.4652105]\,M_3'' & - [8.6908855]\,M_3''' & & = 0
\end{array}
$$

As before, only eight independent relations are afforded by these equations. They may be put in the form

$$
\begin{array}{lll}
M' = -[0.4056419]\,M & M_1' = [9.8713259]\,M_1 & M_2''' = -[9.0766268]\,M_2'' \\
M'' = [8.9288474]\,M & M_1'' = [0.0810458]\,M_1 & \\
M''' = [8.1539175]\,M & M_1''' = [0.5191141]\,M_1 & M_3''' = [0.7743250]\,M_3''
\end{array}
$$

The equations furnished by the initial conditions are

$$
\begin{array}{lr}
M \sin\beta + M_1 \sin\beta_1 & = -[7.5965608] \\
-[0.4056419]\,M \sin\beta + [9.8713259]\,M_1 \sin\beta_1 & = [8.1342320] \\
M \cos\beta + M_1 \cos\beta_1 & = [7.6194702] \\
-[0.4056419]\,M \cos\beta + [9.8713259]\,M_1 \cos\beta_1 & = -[7.9510788] \\
[8.9288474]\,M \sin\beta + [0.0810458]\,M_1 \sin\beta_1 + M_2'' \sin\beta_2 + M_3'' \sin\beta_3 & = -[8.1322283] \\
[8.1539175]\,M \sin\beta + [0.5191141]\,M_1 \sin\beta_1 - [9.0766268]\,M_2'' \sin\beta_2 + [0.7743250]\,M_3'' \sin\beta_3 & = -[7.4662348] \\
[8.9288474]\,M \cos\beta + [0.0810458]\,M_1 \cos B_1 + M_2'' \cos\beta_2 + M_3'' \cos\beta_3 & = [8.0697881] \\
[8.1539175]\,M \cos\beta + [0.5191141]\,M_1 \cos\beta_1 - [9.0766268]\,M_2'' \cos\beta_2 + [0.7743250]\,M_3'' \cos\beta_3 & = -[8.0905661]
\end{array}
$$

From these are derived the following:

$$
\begin{array}{llll}
M \sin\beta = -[7.7020513] & M_1 \sin\beta_1 = [7.0358017] & M_2'' \sin\beta_2 = -[8.1170848] & M_3'' \sin\beta_3 = -[7.1290140] \\
M \cos\beta = [7.5633163] & M_1 \cos\beta_1 = [6.7032913] & M_2'' \cos\beta_2 = [8.1114455] & M_3'' \cos\beta_3 = -[7.3225056]
\end{array}
$$

whence we derive

$$
\begin{array}{llll}
M = [7.7940950] & M_1 = [7.0783152] & M_2'' = [8.2647984] & M_3'' = [7.3971490] \\
\beta = 306^\circ 0' 0''.91 & \beta_1 = 65^\circ 3' 35''.00 & \beta_2 = 314^\circ 37' 40''.85 & \beta_3 = 212^\circ 38' 20''.11
\end{array}
$$

Then if, for brevity, we adopt the notation

$$
\begin{array}{ll}
\chi = 306^\circ\ 0'\ 0''.91 - 25''.921343t\,, & \chi_1 = 65^\circ\ 3'35''.00 - 0''.453700t\,, \\
\chi_2 = 314\ 37\ 40\ .85 - 3\ .107369t\,, & \chi_3 = 212\ 38\ 20\ .11 - 0\ .611574t\,,
\end{array}
$$

we shall have

$$
\begin{array}{lll}
\text{I}\ \sin\theta & = & 0.00622436 \sin\chi + 0.00119761 \sin\chi_1 \\
\text{I}\ \cos\theta & = & 0.00622436 \cos\chi + 0.00119761 \cos\chi_1 \\
\text{I}'\ \sin\theta' & = & -0.01583933 \sin\chi + 0.00089051 \sin\chi_1 \\
\text{I}'\ \cos\theta' & = & -0.01583933 \cos\chi + 0.00089051 \cos\chi_1 \\
\text{I}''\ \sin\theta'' & = & 0.00052838 \sin\chi + 0.00144231 \sin\chi_1 + 0.01839917 \sin\chi_2 + 0.00246054 \sin\chi_3 \\
\text{I}''\ \cos\theta'' & = & 0.00052838 \cos\chi + 0.00144231 \cos\chi_1 + 0.01839917 \cos\chi_2 + 0.00246054 \cos\chi_3 \\
\text{I}'''\ \sin\theta''' & = & 0.00008872 \sin\chi + 0.00395758 \sin\chi_1 - 0.00219495 \sin\chi_2 + 0.01463372 \sin\chi_3 \\
\text{I}'''\ \cos\theta''' & = & 0.00008872 \cos\chi + 0.00395758 \cos\chi_1 - 0.00219495 \cos\chi_2 + 0.01463372 \cos\chi_3
\end{array}
$$

With the sixteen equations the values of the several variables have been computed and inserted in the accompanying table at intervals of a century of Julian years.

Comparison of the numbers of the table has been made with the values from the series in powers of the time

	e	ω	i	θ	e'	ω'	i'	θ'	e''	ω''	i''	θ''	e'''	ω'''	i'''	θ'''
		″	″	″		″	″	″		″	″	″		″	″	″
900	−919	−31.9	+0.82	− 9	+3283	−99.7	−1.19	−38.9	+68	+1.9	−0.15	+ 4.2	+ 4	+0.9	+0.20	+6.1
1400	−163	−6.6	−0.20	− 1	+ 697	−18.5	−0.29	−10.4	+22	+0.4	+0.07	− 1.2	0	+0.2	+0.10	+3.2
1900	0	0	0	0	− 6	0	0	− 0.1	0	0	+0.15	− 0.1	0	+0.3	0	−0.2
2400	− 36	−3.7	−0.20	−11	+ 459	− 6.3	−0.27	−12.4	− 7	0	+0.07	+ 5.7	+ 4	+0.4	−0.10	−3.8
2900	+102	−9.5	−0.79	−53	+1380	− 2.2	−1.04	−54.2	−29	−0.6	−0.12	+13.9	+10	+1.4	−0.21	−7.4

Jupiter.

Date	e		ω		i	θ	
900	0.04665491		10°39′42″.1		20′13″.21	324° 5′ 1″	
		+17132		+711″.0			−2695″
1000	4682623		10 51 33 .1		20 10 .22	323 20 6	
		17063		717 .5			2702
1100	4699686		11 3 30 .6		20 7 .24	322 35 4	
		16995		723 .8			2709
1200	4716681		11 15 34 .4		20 4 .27	321 49 55	
		16921		730 .1			2716
1300	4733602		11 27 44 .5		20 1 .31	321 4 39	
		16848		736 .2			2723
1400	4750450		11 40 0 .7		19 58 .35	320 19 16	
		16778		742 .4			2730
1500	4767228		11 52 23 .1		19 55 .41	319 33 46	
		16701		748 .4			2736
1600	4783929		12 4 51 .5		19 52 .48	318 48 10	
		16623		754 .3			2744
1700	4800552		12 17 25 .8		19 49 .55	318 2 26	
		16544		760 .2			2751
1800	4817096		12 30 6 .0		19 46 .64	317 16 35	
		16464		766 .0			2757
1900	4833560		12 42 52 .0		19 43 .75	316 30 38	
		16383		771 .6			2765
2000	4849943		12 55 43 .6		19 40 .86	315 44 33	
		16301		777 .3			2772
2100	4866244		13 8 40 .9		19 37 .99	314 58 21	
		16217		782 .7			2779
2200	4882461		13 21 43 .6		19 35 .13	314 12 2	
		16133		788 .2			2786
2300	4898594		13 34 51 .8		19 32 .27	313 25 36	
		16047		793 .5			2793
2400	4914641		13 48 5 .3		19 29 .46	312 39 3	
		15959		798 .8			2800
2500	4930600		14 1 24 .1		19 26 .64	311 52 23	
		15871		803 .9			2807
2600	4946471		14 14 48 .0		19 23 .85	311 5 36	
		15778		809 .1			2815
2700	4962249		14 28 17 .1		19 21 .07	310 18 41	
		15690		814 .1			2822
2800	4977939		14 41 51 .2		19 18 .30	309 31 39	
		+15593		+819 .0			−2831
2900	0.04993532		14 55 30 .2		19 15 .55	308 44 28	

Saturn.

Date	e'		ω'		i'	θ'	
900	0.05925956		85°30′41″.5		55′40″.25	130°16′39″.8	
		−32981		+1986″.5			−2532″.4
1000	5892975		86 3 48 .0		55 42 .30	129 34 27 .4	
		33152		1990 .9			2530 .9
1100	5859823		86 36 58 .9		55 44 .34	128 52 16 .5	
		33324		1995 .3			2529 .5
1200	5826499		87 10 14 .2		55 46 .36	128 10 7 .0	
		33496		1999 .8			2528 .0
1300	5793003		87 43 34 .0		55 48 .38	127 27 59 .0	
		33671		2004 .4			2526 .5
1400	5759332		88 16 58 .4		55 50 .38	126 45 52 .5	
		33840		2009 .1			2525 .2
1500	5725492		88 50 27 .5		55 52 .38	126 3 47 .3	
		34006		2014 .1			2523 .6
1600	5691486		89 24 1 .6		55 54 .34	125 21 43 .7	
		34172		2018 .9			2522 .2
1700	5657314		89 57 40 .5		55 56 .31	124 39 41 .5	
		34339		2024 .0			2520 .9
1800	5622975		90 31 24 .5		55 58 .25	123 57 40 .6	
		34501		2029 .1			2519 .4
1900	5588474		91 5 13 .6		56 0 .19	123 15 41 .2	
		34662		2034 .4			2518 .2
2000	5553812		91 39 8 .0		56 2 .11	122 33 43 .0	
		34821		2039 .9			2516 .7
2100	5518991		92 13 7 .9		56 4 .01	121 51 46 .3	
		34981		2045 .4			2515 .3
2200	5484010		92 47 13 .3		56 5 .91	121 9 51 .0	
		35138		2051 .1			2513 .9
2300	5448872		93 21 24 .4		56 7 .78	120 27 57 .1	
		35291		2056 .9			2512 .9
2400	5413581		93 55 41 .3		56 9 .64	119 46 4 .2	
		35445		2062 .9			2511 .4
2500	5378136		94 30 4 .2		56 11 .49	119 4 12 .8	
		35599		2069 .0			2510 .1
2600	5342537		95 4 33 .2		56 13 .31	118 22 22 .7	
		35749		2075 .3			2508 .8
2700	5306788		95 39 8 5		56 15 .12	117 40 33 .9	
		35896		2081 .6			2507 .6
2800	5270892		96 13 50 .1		56 16 .91	116 58 46 .3	
		36043		2088 .2			−2506 .2
2900	0.05234849		96 48 38 .3		56 18 .70	116 17 0 .1	

Uranus.

Date	e''		ω''		i''	θ''	
900	0.04660145		170°42′57″.6		61′37″.47	311°59′ 0″.3	
		−2797		+315″.5			−391″.9
1000	4657348		170 48 13 .1		61 37 .76	311 52 28 .4	
		2780		314 .7			391 .9
1100	4654568		170 53 27 .8		61 38 .04	311 45 56 .5	
		2768		314 .1			391 .9
1200	4651800		170 58 41 .9		61 38 .30	311 39 24 .6	
		2756		313 .3			391 .8
1300	4649044		171 3 55 .2		61 38 .55	311 32 52 .8	
		2743		312 .5			391 .7
1400	4646301		171 9 7 .7		61 38 .78	311 26 21 .1	
		2727		312 .0			391 .7
1500	4643574		171 14 19 .7		61 39 .00	311 19 49 .4	
		2713		311 .3			391 .7

Uranus — (continued).

Date	e'''		ω'''		i'''	θ'''	
1600	4640861		171 19 31 .0		61 39 .21	311 13 17 .7	
		2699		310 .6			391 .6
1700	4638162		171 24 41 .6		61 39 .40	311 6 46 .1	
		2683		310 .0			391 .5
1800	4635479		171 29 51 .6		61 39 .59	311 0 14 .6	
		2667		309 .2			391 .4
1900	4632812		171 35 0 .8		61 39 .76	310 53 43 .2	
		2654		308 .5			391 .4
2000	4630158		171 40 9 .3		61 39 .91	310 47 11 .8	
		2640		308 .1			391 .3
2100	4627518		171 45 17 .4		61 40 .06	310 40 40 .5	
		2624		307 .5			391 .2
2200	4624894		171 50 24 .9		61 40 .19	310 34 9 .3	
		2610		306 .7			391 .1
2300	4622284		171 55 31 .6		61 40 .31	310 27 38 .2	
		2594		306 .0			391 .0
2400	4619690		172 0 37 .6		61 40 .42	310 21 7 .2	
		2577		305 .6			390 .8
2500	4617113		172 5 43 .2		61 40 .52	310 14 36 .4	
		2561		304 .9			390 .8
2600	4614552		172 10 48 .1		61 40 .60	310 8 5 .6	
		2545		304 .3			390 .6
2700	4612007		172 15 52 .4		61 40 .67	310 1 35 .0	
		2528		303 .7			390 .4
2800	4609479		172 20 56 .1		61 40 .73	309 55 4 .6	
		−2512		+303 .0			−390 .2
2900	0.04606967		172 25 59 .1		61 40 .78	309 48 34 .4	

Neptune.

Date	e'''		ω'''		i'''	θ'''	
900	0.00890705		46°27′55″.0		43′33″.78	193°34′48″.0	
		+603		+80″.6			−78″.4
1000	891308		46 29 15 .6		43 33 .56	193 33 29 .6	
		603		81 .1			78 .5
1100	891911		46 30 36 .7		43 33 .34	193 32 11 .1	
		604		81 .5			78 .7
1200	892515		46 31 58 .2		43 33 .12	193 30 52 .4	
		605		82 .0			78 .9
1300	893120		46 33 20 .2		43 32 .90	193 29 33 .5	
		605		82 .6			79 .0
1400	893725		46 34 42 .8		43 32 .69	193 28 14 .5	
		604		83 .2			79 .1
1500	894329		46 36 6 .0		43 32 .47	193 26 55 .4	
		604		83 .6			79 .3
1600	894933		46 37 29 .6		43 32 .26	193 25 36 .1	
		604		84 .0			79 .5
1700	895537		46 38 53 .6		43 32 .04	193 24 16 .6	
		604		84 .3			79 .6
1800	986141		46 40 17 .9		43 31 .83	193 22 57 .0	
		605		85 .3			79 .7
1900	896746		46 41 43 .2		43 31 .62	193 21 37 .3	
		605		85 .9			79 .9
2000	897351		46 43 9 .1		43 31 .40	193 20 17 .4	
		603		86 .1			80 .1

Neptune — (*continued*).

Date	e'''		ω'''		i'''	θ'''	
2100	897954		46 44 35 .2		43 31 .19	193 18 57 .3	
		605		86 .8			80 .2
2200	898559		46 46 2 .0		43 30 .98	193 17 37 .1	
		605		87 .2			80 .4
2300	899164		46 47 29 .2		43 30 .77	193 16 16 .7	
		603		87 .8			80 .6
2400	899767		46 48 57 .0		43 30 .56	193 14 56 .1	
		604		88 .3			80 .8
2500	900371		46 50 25 .3		43 30 .36	193 13 35 .3	
		605		88 .7			80 .9
2600	900976		46 51 54 .0		43 30 .15	193 12 14 .4	
		603		89 .3			81 .0
2700	901579		46 53 23 .3		43 29 .94	193 10 53 .4	
		603		89 .7			81 .3
2800	902182		46 54 53 .0		43 29 .73	193 9 32 .1	
		+604		+90 .3			−81 .4
2900	0.00902786		46 56 23 .3		43 29 .53	193 8 10 .7	

THE PROPER-MOTION OF Ci_2 1536,

BY J. G. PORTER.

In *A.N.* 4635 there is a note by Professor SLOCUM on the proper motion and parallax of a faint star. This note is repeated in the article by FREDERICK SLOCUM and S. A. MITCHELL on stellar parallaxes in the July number of the *Astrophysical Journal*.

The star in question has been observed several times at this observatory. It is number 1536 of Publication No. 13, and was observed four times during the past summer by Dr. ELLIOTT SMITH.

From these determinations I find the proper-motion as follows:

	Epoch.	R.A. 1910.0	Obs.	Decl. 1910.0
		h m s		° ′ ″
Ci_2	1898.2	17 33 50.40	3	+18 36 58.6
Ci	1913.6	51.42	4	37 14.7
		P.M. $+0^s.066$		$+1''.05$

Professor SLOCUM's motion is quite close, but the declination given by him appears to be 40″ too small.

Cincinnati Observatory, Sept. 23, 1913.

HALLEY'S COMET,

ELEMENTS BASED UPON OBDERVATIONS TAKEN AFTER PERIHELION PASSAGE OF APRIL 19.69 = 1910, AND EPHEMERIS FOR NEXT OPPOSITION,

BY F. E. SEAGRAVE.

$$
\begin{aligned}
E &= \text{June } 24.50 = 1910. \quad \text{G.M.T.} \\
M &= 0^\circ\ 51'\ 10''.67 \\
\pi &= 168^\circ\ 58'\ 27''.96 \\
\Omega &= 57^\circ\ 16'\ 12''.11 \\
i &= 162^\circ\ 12'\ 41''.96 \\
\text{Log } e &= 9.985550. \\
\text{Log } a &= 1.254001. \\
\text{Log } q &= 9.768825. \\
\mu &= 46''.6667.
\end{aligned}
$$

$$
\left.\begin{aligned}
x &= r\,(9.98516)\ \sin(\ 34^\circ\ \ 1'\ \ 9''.01+u) \\
y &= r\,(9.98848)\ \sin(127^\circ\ 34'\ 57''.00+u) \\
z &= r\,(9.53554)\ \sin(\ 77^\circ\ 18'\ 47''.80+u)
\end{aligned}\right\}\ \text{CONSTANTS.}
$$

Heliocentric and geocentric positions in orbit at epoch = June 24.50 = 1910. G.M.T.

$$
\left.\begin{aligned}
E &= 17^\circ\ 44'\ \ 8''.02 \\
v &= 100^\circ\ 50'\ 48''.38 \\
\lambda &= 205^\circ\ 58'\ 42''.02 \\
\beta &= -9^\circ\ 27'\ 39''.59 \\
\text{Log } r &= 0.149923 \\
u &= 212^\circ\ 33'\ \ 4''.38
\end{aligned}\right\}\ \text{HELIOCENTRIC.}
$$

$$
\left.\begin{aligned}
l &= 162^\circ\ 38'\ 31''.88 \\
b &= -9^\circ\ 41'\ 48''.31 \\
\text{Log } \Delta &= 0.139333 \\
\text{Log } \rho &= 0.133385
\end{aligned}\right\}\ \text{GEOCENTRIC.}
$$

EPHEMERIS.

1914	R.A.	Dec.	Log r	Log Δ
	h m s	° ′ ″		
Feb. 1	= 9 33 19.	−5 52 28.	1.07960	1.04563
5	= 9 31 50.	−5 46 9.	1.08040	1.04587
9	= 9 30 21.	−5 39 13.	1.08120	1.04633
☍	=			
13	= 9 28 52.	−5 31 52.	1.08200	1.04697
17	= 9 27 26.	−5 24 12.	1.08280	1.04782
21	= 9 25 57.	−5 15 38.	1.08358	1.04884

The above ephemeris was prepared for Prof. BARNARD. The Yerkes and Mount Wilson Observatories are much interested in photographing comets at great distances from the Sun and Earth. As the comet will be 11.20 units from the Earth next February, I do not think there will be any hope of finding it.

OBSERVATIONS OF COMET 1913 d,

MADE WITH THE FILAR MICROMETER OF THE 433MM EQUATORIAL OF LA PLATA OBSERVATORY,

BY W. J. HUSSEY.

1913	La Plata M.T.	∗	Comp.	☄ − ∗ $\Delta\alpha$	☄ − ∗ $\Delta\delta$	☄'s Apparent α	☄'s Apparent δ	Log $p'\Delta$ for α	Log $p'\Delta$ for δ
	h m s			s	′ ″	h m s	° ′ ″		
Sept. 26	10 29 7.3	1	8 , 8	−13.17	−0 15.6	21 54 18.36	−2 34 27.4	9.0654	0.6709n
27	8 15 26.4	3	8 , 8	− 4.00	−0 49.2	21 51 20.61	−1 48 17.5	9.1682n	0.6802n
28	11 25 38.1	5	8 , 8	−10.01	−1 32.8	21 47 40.24	−0 50 17.9	9.4042	0.6917n
30	9 10 25.4	7	8 , 8	− 9.05	−2 16.7	21 41 34.75	+0 49 1.2	8.8718	0.7083n
Oct. 1	10 48 43.4	8	8 , 8	− 3.79	−0 56.5	21 38 20.63	+1 43 4.4	9.3518	0.7157n
2	12 21 48.4	10	8 , 8	− 0.14	−4 58.8	21 35 7.24	+2 37 51.5	9.5852	0.7162n
3	8 36 52.1	12	8 , 8	− 1.44	−4 1.3	21 32 37.78	+3 21 17.4	8.1900n	0.7341n
4	8 23 51.1	15	8 , 8	− 0.12	+1 30.3	21 29 44.61	+4 12 4.2	8.4543n	0.7421n
5	11 24 48.3	17	8 , 8	−19.20	−2 37.4	21 26 31.44	+5 9 25.5	9.5242	0.7369n

Mean Places of Comparison Stars.

∗	α 1913.0	Red. to App. Place	δ 1913.0	Red. to App. Place	Authority
	h m s	s	° ′ ″	s	
1	21 54 28.07	+3.46	−2 34 27.7	+15.9	Connected with Star 2.
2	21 58 48.83	+3.46	−2 34 32.8	+16.2	A.G. Strassburg 7695.
3	21 51 21.19	+3.42	−1 47 44.1	+15.8	Connected with Star 4.
4	21 54 53.16	+3.43	−1 47 49.6	+16.0	A.G. Strassburg 7678.
5	21 47 46.88	+3.37	−0 49 0.8	+15.7	Connected with Star 6.
6	21 50 52.06	+3.38	−0 53 53.4	+15.9	A.G. Nicolajew 5529.
7	21 41 40.51	+3.29	+0 51 2.3	+15.6	A.G. Nicolajew 5509.
8	21 38 21.17	+3.25	+1 43 45.3	+15.6	Connected with Star 9.
9	21 35 20.23	+3.24	+1 44 44.4	+15.4	A.G. Albany 7567.
10	21 35 4.17	+3.21	+2 42 34.7	+15.6	Connected with Star 11.
11	21 36 9.29	+3.21	+2 47 17.5	+15.7	A.G. Albany 7573.
12	21 32 36.04	+3.18	+3 25 3.0	+15.7	Connected with Star 13.
13	21 32 36.06	+3.18	+3 30 48.6	+15.7	Connected with Star 14, (is B.D. +3°4584).
14	21 36 25.19	+3.19	+3 30 5.9	+16.0	A.G. Albany 7576.
15	21 29 41.59	+3.14	+4 10 18.2	+15.7	Connected with Star 16.
16	21 28 48.58	+3.14	+4 10 38.8	+15.7	A.G. Albany 7535.
17	21 26 47.53	+3.11	+5 11 47.1	+15.8	A.G. Albany 7527.

This comet was discovered by Mr. PAUL T. DELAVAN, with the Zeiss Comet Seeker of this observatory on the evening of September 26, 1913. The close agreement of the observed path with that which may be computed by slightly changing the elements of comet Westphal leaves little doubt that this is a return of that comet.

The above observations have been reduced by Mr. B. H. DAWSON. He has also connected the comparison stars, except No. 3.

Observatorio Astronómico, La Plata, Argentina, October 9, 1913.

A CONTROL FOR LEAST SQUARE SOLUTIONS.

By JOSEPH F. RITT.

The solution, by the method of least squares, of the system of linear equations

$$(1)\begin{cases} a_1x_1 + a_2x_2 + \ .\ .\ . + a_px_p + \ .\ .\ . + a_rx_r + \mathrm{A} = 0 \\ b_1x_1 + b_2x_2 + \ .\ .\ . + b_px_p + \ .\ .\ . + b_rx_r + \mathrm{B} = 0 \\ .\quad .\quad .\quad .\quad .\quad .\quad .\quad . \\ n_1x_1 + n_2x_2 + \ .\ .\ . + n_px_p + \ .\ .\ . + n_rx_r + \mathrm{N} = 0, \end{cases}$$

where r exceeds n, is most conveniently effected by expressing the unknowns in the form

$$(2)\qquad x_p = a_p\,\mathrm{L}_1 + b_p\,\mathrm{L}_2 + \ .\ .\ .\ . + n_p\,\mathrm{L}_n\,,$$

where the correlates $\mathrm{L}_1, \mathrm{L}_2, \ldots, \mathrm{L}_n$, are determined by the normal equations

$$(3)\begin{cases} [a\,a]\,\mathrm{L}_1 + [a\,b]\,\mathrm{L}_2 + \ .\ .\ . + [a\,n]\,\mathrm{L}_n + \mathrm{A} = 0 \\ [a\,b]\,\mathrm{L}_1 + [b\,b]\,\mathrm{L}_2 + \ .\ .\ . + [b\,n]\,\mathrm{L}_n + \mathrm{B} = 0 \\ .\quad .\quad .\quad .\quad .\quad .\quad . \\ [a\,n]\,\mathrm{L}_1 + [b\,n]\,\mathrm{L}_2 + \ .\ .\ . + [n\,n]\,\mathrm{L}_n + \mathrm{N} = 0. \end{cases}$$

This normal system may then be treated by the method of elimination introduced by GAUSS, which reduces it to the form

$$(4)\begin{cases} \mathrm{L}_1 = \beta_1\,\mathrm{L}_2 + \gamma_1\,\mathrm{L}_3 + \ .\ .\ . + \mu_1\,\mathrm{L}_{n-1} + \nu_1\,\mathrm{L}_n + \pi_1 \\ \mathrm{L}_2 = \qquad\quad \gamma_2\,\mathrm{L}_3 + \ .\ .\ . + \mu_2\,\mathrm{L}_{n-1} + \nu_2\,\mathrm{L}_n + \pi_2 \\ .\quad .\quad .\quad .\quad .\quad .\quad .\quad . \\ \mathrm{L}_{n-1} = \qquad\qquad\qquad\qquad\qquad \nu_{n-1}\mathrm{L}_n + \pi_{n-1} \\ \mathrm{L}_n \ = \qquad\qquad\qquad\qquad\qquad\qquad\quad \pi_n\,. \end{cases}$$

For the control of the computation of the several quantities $\beta, \gamma, \ldots, \mu, \nu, \pi$, there exists the well known check which eliminates almost all possibility of error in their calculation. However, an error made in any of the substitutions indicated by equations (4) will affect all substitutions which follow it, and will not be detected until the unknowns $x_1, x_2, \ldots, x_r$, are computed by the relation (2) and substituted in the original equations (1). If these are not satisfied we know only that an error exists, without knowing its location, and when the number of equations to be solved reaches forty or fifty, considerable labor will be rendered worthless by any inaccurracy committed in the computation of the correlates by equations (4). It is customary, in such cases, to duplicate this portion of the solution.

We propose to indicate, in this paper, a control of the computation in question, which, while perhaps no more rapid than duplication, possesses a sufficient number of advantages to recommend its use where a duplication would ordinarily be executed. Writing

$$s_p = a_p + b_p + \ .\ .\ .\ .\ .\ . + n_p\,,$$
$$\text{and } \mathrm{S} = \mathrm{A} + \mathrm{B} + \ .\ .\ .\ .\ .\ . + \mathrm{N}\,,$$

let us introduce into our system of equations (1), the equation

$$s_1\,x_1 + s_2\,x_2 + \ .\ .\ . + s_p\,x_p + \ .\ . + s_r\,x_r + \mathrm{S} = 0.$$

Now, if we associate with our equations, the correlates $l_1, l_2, \ldots, l_n, l_s$, we obtain the new normal system

$$\left.\begin{array}{l} [a\,a]\,l_1 + [a\,b]\,l_2 + \ .\ .\ . + [a\,n]\,l_n + [a\,s]l_s + \mathrm{A} = 0 \\ [a\,b]\,l_1 + [b\,b]\,l_2 + \ .\ .\ . + [b\,n]\,l_n + [b\,s]l_s + \mathrm{B} = 0 \\ .\quad .\quad .\quad .\quad .\quad .\quad .\quad . \\ [a\,n]\,l_1 + [b\,n]\,l_2 + \ .\ .\ . + [n\,n]\,l_n + [n\,s]l_s + \mathrm{N} = 0 \\ [a\,s]\,l_1 + [b\,s]\,l_2 + \ .\ .\ . + [n\,s]\,l_n + [s\,s]l_s + \mathrm{S} = 0, \end{array}\right\}(5)$$

of which, the last equation is the sum of all the others. This linear dependence will, of course, cause a degeneration of the solution of the system, and an indeterminate value of l_s will result. However, we can distribute the coefficient of l_s in each of the first n equations of (5), and write

$$\begin{array}{l} [a\,a]\,(l_1+l_s) + [a\,b]\,(l_2+l_s) + \ .\ .\ . + [a\,n]\,(l_n+l_s) + \mathrm{A} = 0 \\ [a\,b]\,(l_1+l_s) + [b\,b]\,(l_2+l_s) + \ .\ .\ . + [b\,n]\,(l_n+l_s) + \mathrm{B} = 0 \\ .\quad .\quad .\quad .\quad .\quad .\quad .\quad . \\ [a\,n]\,(l_1+l_s) + [b\,n]\,(l_2+l_s) + \ .\ .\ . + [n\,n]\,(l_n+l_s) + \mathrm{N} = 0, \end{array}$$

from which it is evident, that whatever value be assigned to l_s,

$$l_p + l_s = \mathrm{L}_p\,. \qquad (6)$$

The reduction of equations (5) to a system similar to (4), involves hardly any additional labor. The computation of the l's can then be carried on simultaneously with that of the L's, and the check formula (6) applied at each step. Thus, the control, besides furnishing an independent computation of the correlates, is further superior to duplication in that it permits an immediate comparison of each L with its corresponding l, and enables a single computer to execute the entire solution. It also eliminates errors resulting from the presence of doubtful figures in the quantities $\beta, \gamma, \ldots, \mu, \nu, \pi$, a source of frequent inaccuracy, to which the duplicator is as susceptible as the original computer. It fails, however, should equal errors be committed in the computations of an L and its corresponding l.

Deferring, for the moment, the application of the foregoing theory to a practical example, we shall consider the solution of a system of equations where the number of conditions exceeds the number of unknowns. Our check process is also applicable here, but as solutions of this kind seldom involve many normal equations, little occasion will be found for its use.

Take, then, the system

$$
\begin{aligned}
a_1 x_1 + b_1 x_2 + \ldots\ldots + n_1 x_n + K_1 &= 0 \\
a_2 x_1 + b_2 x_2 + \ldots\ldots + n_2 x_n + K_2 &= 0 \\
\ldots\ldots\ldots\ldots\ldots \\
a_r x_1 + b_r x_2 + \ldots\ldots + n_r x_n + K_r &= 0\,,
\end{aligned}
$$

whose normal equations have an array of coefficients similar to that of equations (3), but absolute terms $[a\mathrm{K}]$, $[b\ \mathrm{K}]$, ... , $[n\ \mathrm{K}]$, and compare it with the system

$$
\begin{aligned}
a_1 y_1 + b_1 y_2 + \ldots\ldots + n_1 y_n + s_1 y_s + K_1 &= 0 \\
a_2 y_1 + b_2 y_2 + \ldots\ldots + n_2 y_n + s_2 y_s + K_2 &= 0 \\
\ldots\ldots\ldots\ldots\ldots \\
a_r y_1 + b_r y_2 + \ldots\ldots + n_r y_n + s_r y_s + K_r &= 0\,,
\end{aligned}
$$

where

$$s_p = a_p + b_p + \ldots\ldots + n_p\,.$$

The normal equations of this system will have an array of coefficients similar to that of equations (5), and absolute terms $[a\ \mathrm{K}]$, $[b\ \mathrm{K}]$, ... , $[n\ \mathrm{K}]$, $[s\ \mathrm{K}]$. They will also be linearly dependent, but whatever value be chosen for y_s, we should find

$$x_p = y_p + y_s\,.$$

To illustrate the check process by a practical example, we shall consider a simple system of three normal equations, with the array of terms

x_1	x_2	x_3	K	Σ
1	+4	+7	+3	+15
	2	−3	+5	+ 8
		4	−3	+ 5

where the column headed Σ contains the terms necessary for a control of the reduction of the equations to the form (4).

Whatever method be employed for the reduction we shall arrive at the following equations, by which each unknown is expressed in terms of those which follow it:

	Σ	C
$x_1 = -4.000\, x_2 - 7.000\, x_3 - 3.000$	−15.000	−12.000
$x_2 = -2.214\, x_3 - 0.500$	− 3.714	− 3.214
$x_3 = +\,0.359$	− 0.641	− 1.000

The column headed C contains the difference between each term of the column Σ and the absolute term which stands to its left. If, now, we choose 0.333 for the value of an indeterminate y_s and compute y_1, y_2, and y_3 from the equations

$$
\begin{aligned}
y_1 &= -4.000\, y_2 - 7.000\, y_3 - 12.000\, y_s - 3.000 \\
y_2 &= -2.214\, y_3 - 3.214\, y_s - 0.500 \\
y_3 &= -1.000\, y_s + 0.359\,,
\end{aligned}
$$

we shall find each y to be less by 0.333 than its corresponding x.

It will be seen, upon examining the above problem that the operative principle of the check is contained in the equation

$$
\begin{aligned}
L_1 = {} & \beta_1(L_2 - L_s) + \gamma_1(L_3 - L_s) + \ldots\ldots + \mu_1(L_{n-1} - L_s) \\
& + \nu_1(L_n - L_s) + (\beta_1 + \gamma_1 + \ldots\ldots + \mu_1 + \nu_1 - 1)\, L_s + L_s,
\end{aligned}
$$

which might have been made the starting point of our discussion.

Washington, D.C.

NOTICE.

Mr. J. VAN DER BILT, astronomer at the University Observatory, Utrecht (Holland) has undertaken the definitive reduction of all available observations of *R. Sagittæ*, *V. Vulpeculæ* and *RV. Tauri*, and would be very glad to have copies of any unpublished observations, in such detail, that they can be reduced by a normal photometric light scale.

CONTENTS.

ACTING EDITOR, BENJAMIN BOSS, ALBANY, N.Y.; ASSOCIATE EDITORS: SETH C. CHANDLER AND GEORGE W. HILL, DIRECTORS OF THE GOULD, FUND OF THE NATIONAL ACADEMY OF SCIENCES AND PROF. ERNEST W. BROWN, OF YALE UNIVERSITY. PUBLISHED BY THE DUDLEY OBSERVATORY, ALBANY, N.Y., U.S.A., TO WHICH ALL COMMUNICATIONS SHOULD BE ADDRESSED. PRICE, $5.00 THE VOLUME. PRESS OF THOS. P. NICHOLS & SON CO., LYNN, MASS. *Closed Nov.* 29, 1913.

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A STUDY OF NEBULÆ.

By E. A. Fath.

This paper is based on a study of the nebulæ found on plates of one hundred and thirty-nine regions of the sky taken with the sixty-inch reflector of the Mt. Wilson Solar Observatory during the years 1909-12. The plates form a portion of the work on the "Selected Areas" of Kapteyn* undertaken by that observatory and cover the central portion of each area that can be reached satisfactorily from the latitude of Mt. Wilson. This includes all the regions from the north celestial pole to 15° south declination. Lumière Sigma plates, measuring six and one-half by eight and one-half inches, were used. The exposed area of each plate contained 1.88 square degrees. The length of the exposures was one hour each. For various reasons it was necessary to take a second plate of about twenty-five per cent. of the regions. Of the final plates seventeen were taken by Mr. Harold D. Babcock and the remaining one hundred and twenty-two by the writer.

At first it did not seem advisable to publish the list of new nebulæ because of the fact that duplicate plates were available for only one quarter of the areas. However, in order to test matters, the nebulæ were marked independently on such plates before they were compared. It was then found that so few spurious objects had been marked, particular pains being taken to exclude everything doubtful, that the list as a whole seemed entitled to considerable confidence. It is accordingly submitted below, Table I, with the hope that it may prove to be as reliable as the tests indicated.

In studying the nebulæ on these plates the following points were noted: 1. Right ascension and declination for 1910.0; 2. Dimensions; 3. Position angle of greatest diameter; 4. Brightness. The results given are the mean of two independent measures.

1. The determination of position was made by placing a plate, film down, upon paper which had been ruled so that positions could be read off directly when the position of the central star on the plate was known. The position of this star was in each instance taken from the catalogue of the *Astronomische Gesellschaft*. The positions given are in general correct within one minute of arc. This accuracy was determined from positions of stars, several of which were measured on each plate as checks.

2. The dimensions were obtained with a scale divided in half millimeters, tenths of millimeters being estimated. A lens magnifying four diameters was used while making these measures. The dimensions in millimeters were then transformed into minutes and seconds of arc. Since practically all of the nebulæ were of elliptical outline the major and minor axes were the quantities measured.

3. A straight edge was placed parallel to the major axis and the angle between this and an hour circle was measured to the nearest degree. These angles were then transformed into position angles ranging from 0° to 180°.

4. An effort was made to estimate intrinsic brightness. For small objects this was by no means easy, nevertheless, it is believed that the estimates as a whole are fairly homogeneous. In the tables B = bright and F = faint. Various degrees are expressed by V = very and M = moderately. The order of brightness is as follows: VB, B, MB, MF, F, VF.

Table I gives a list of the new nebulæ that were found. The only column needing explanation is the 4th. Here may be found the dimensions "length by width." When the term stellar is used in the last column it signifies that the nebula was small and looked like a star surrounded by an atmosphere. This probably corresponds to psmbM in the *N. G. C.* In a number of cases the intensity diminished so gradually that no accurate measures of the dimensions could be made and they are accordingly omitted.

Table I.

No.	α	δ	LxW	Elong.	Inten.	Remarks
1	$0^h11^m12^s$	+15°35′	1′21″x11″	94°	MB	3 condensations.
2	0 11 53	+15 41	11″x 5″	161	MF	
3	0 12 12	−14 21	27 x 8	130	F	
4	0 14 40	−14 55	16 x11	113	VF	MB nucleus.
5	0 14 53	−14 14	27 x11	34	F	
6	0 14 53	−14 16	... ...	...	F	Stellar.

* The Plan of Selected Areas. Kapteyn. Groningen, 1906.

No	α	δ	LxW	Elong.	Inten.	Remarks
7	0h22m49s	−30°30′	14″x 3″	100°	F	
8	0 23 09	+30 06	5 x 5	.	MF	Nucleus.
9	0 23 32	+30 19	27 x 3	116	F	
10	0 23 35	−30 42	14 x14	.	F	
11	0 23 50	+30 10	5 x 3		MF	Stellar.
12	0 23 57	+30 13	14 x 5	26	F	
13	0 24 24	+30 31	19 x 3	112	MF	Double.
14	0 24 41	+30 04	8 x 3	.	F	Nucleus stellar.
15	0 25 26	−30 12	8 x 3	113	F	
16	0 25 29	+30 06	5 x 3	0	F	
17	0 13 16	−15 52		.	MF	Stellar.
18	0 48 18	− 0 06	14 x14		F	
19	0 48 20	− 0 08	11 x 8	83	F	
20	0 49 16	− 0 30	5 x 5		MB	Stellar.
21	0 49 31	− 0 26	8 x 5	152	F	
22	0 49 37	+ 0 35	8 x 5	12	F	
23	0 50 21	+ 0 07	14 x 5	113	MF	4 neb.spots with
24	0 50 21	− 0 53	54 x11	75	MF	[star(?)NW end.
25	0 50 54	− 0 33	5 x 3	12	F	
26	0 50 56	− 0 03	8 x 5	135	F	
27	0 51 07	− 0 13	5 x 5	.	MF	Stellar.
28	0 51 09	− 0 09	3 x 3		MF	Stellar.
29	0 51 26	− 0 13	14 x 8	0	F	
30	0 51 29	− 0 08	3 x 3	.	F	Stellar.
31	0 51 32	− 0 12	3 x 3		F	
32	0 51 37	− 0 11	3 x 3		F	
33	0 52 10	+ 0 11			B	Stellar.
34	0 52 51	− 0 17	8 x 5	78	F	
35	1 09 46	15 01	11 x11	.	F	Stellar.
36	1 09 46	−15 08	19 x11	78	F	
37	1 09 51	−14 59	11 x11	.	F	Stellar.
38	1 10 19	−14 27	14 x14		VF	
39	1 10 26	−14 36	11 x11		MF	
40	1 10 26	−14 42	16 x16		VF	
41	1 10 27	−14 56	8 x 5	140	F	
42	1 10 36	−14 09	5 x 5		F	
43	1 10 41	−14 20	14 x 8	140	MB	Stellar.
44	1 10 42	−14 33	14 x14		MB	
45	1 10 54	−14 57	16 x 8	75	F	
46	1 11 00	−14 55	24 x11	141	MF	
47	1 11 03	−14 27	8 x 8		MB	Stellar.
48	1 11 12	−14 30	8 x 8		MF	Stellar.
49	1 11 16	−15 00	19 x11	0	MF	
50	1 11 26	14 18	16 x 8	150	F	
51	1 11 32	−14 39	8 x 3	58	F	
52	1 11 38	−14 52	14 x 5	28	F	
53	1 11 42	−14 51	5 x 5		F	
54	1 11 50	14 14	5 x 5		MB	
55	1 11 53	14 24	27 x 8	121	MF	
56	1 11 59	−14 24	8 x 5	0	MF	Stellar.
57	1 11 59	−14 10	8 x 5	25	MF	
58	1 12 00	14 31	5 x 3	90	MF	
59	1 12 03	−14 22	19 x11	129	F	
60	1 12 10	−14 49	14 x 3	161	F	
61	1 12 11	−15 05	11 x 8	169	MF	Stellar.
62	1 12 14	−14 29	11 x 8	120	F	
63	1 12 19	−14 22	14 x11	90	VF	
64	1 12 36	−14 41	8 x 8		MF	
65	1 12 41	−14 29	19 x 8	150	MB	Stellar.
66	1 12 52	−14 45	8 x 8	.	MF	
67	1 13 04	−14 52	19 x11	147	F	
68	1 13 08	−14 06	8 x 8		F	
69	1h13m10s	−14°30′	11″x11″	°	F	
70	1 13 24	−14 09	11 x11	.	F	
71	1 13 28	−14 11	16 x11	113	MF	Stellar.
72	1 13 28	−14 40	16 x 5	75	F	
73	1 13 48	−14 22	11 x 8	56	MF	Stellar.
74	1 14 18	+15 37	5 x 5		F	
75	1 14 36	−15 00	14 x14		MB	Stellar.
76	1 14 54	−14 23	16 x11	31	MB	Stellar.
77	1 15 06	−15 00	14 x14		VF	
78	1 15 08	−14 12	19 x19		VF	
79	1 15 11	+15 08	3 x 3		F	
80	1 15 33	+15 32	5 x 5		B	Stellar.
81	1 17 23	+15 49	8 x 8		MF	
82	1 17 30	+15 50	22 x 8	29	MF	
83	1 23 57	+30 26	11 x 8		MB	Stellar.
84	1 24 25	+30 10	16 x14	136	F	
85	1 24 28	+30 10	14 x 5	90	F	
86	1 24 49	+30 25	5 x 5	.	F	
87	1 25 08	+29 56	19 x 3	177	F	
88	1 25 41	+29 55	11 x11		F	
89	1 25 41	+30 31	14 x 5	157	MB	
90	1 25 49	+30 22	11 x 5	165	F	
91	1 25 49	+30 27	14 x 8	138	MB	
92	1 25 50	+30 01	27 x 5	138	F	
93	1 25 56	+29 58	14 x11	120	F	
94	1 26 13	+29 59	11 x 5	3	MF	2 nuclei.
95	1 26 25	+29 57	11 x 8	108	F	[s. brighter.
96	1 26 45	+30 02	16 x11	58	MF	Almost certainly
						[spiral.
97	1 35 15	+45 23	5 x 5		B	Stellar.
98	1 47 57	+ 0 54	16 x11	17	F	
99	1 48 37	+ 0 36	11 x 8	153	F	
100	1 49 14	+ 0 17	5 x 5		VF	
101	1 49 29	+ 0 31	14 x11	90	F	
102	1 49 37	+ 0 43	5 x 5	.	F	
103	1 49 45	+ 0 11	5 x 3	50	F	
104	1 51 13	+ 0 16	5 x 3	90	MB	
105	1 51 33	− 0 11	5 x 5		VF	
106	2 14 10	+14 51	14 x11	36	F	
107	2 16 19	−14 40	24 x 5	8	F	
108	2 17 07	+15 09	2 x 2		MB	Stellar.
109	2 17 17	−14 42	5 x 5		F	Stellar.
110	2 17 35	−14 30	. . .		F	Stellar.
111	2 26 49	+30 13	14 x11		MB	Stellar.
112	2 26 57	+30 45	27 x 8	46	F	
113	2 27 26	+30 25	16 x11	.	MB	Stellar.
114	2 27 35	+30 21	27 x14	12	MB	Looks very much
						like neb. star.
115	2 27 47	+30 07	19 x 8	130	MB	
116	2 28 05	+30 27	27 x 8	147	B	
117	2 28 06	+30 28	24 x14	6	MF	
118	2 28 10	+29 48	16 x11	17	MB	
119	2 28 29	+29 47	22 x22	. . .	MB	One arm of spiral
120	2 28 32	+29 46	19 x 8	105	F	[vis.
121	2 29 25	+30 21	14 x 8	110	F	
122	2 29 30	+29 46	11 x 5	170	MF	
123	2 29 30	+29 50	19 x19	.	MB	Some detail.
124	2 29 52	+29 59	5 x 5	. . .	MF	[Spiral (?)
125	2 30 42	+30 08	14 x 8	39	VF	
126	2 32 52	+30 27	54 x41	17	VF	One arm of spiral
						[vis.
127	2 51 19	+ 0 20	1′00 x11	145	MF	Nucleus.

No.	α	δ	LxW	Elong.	Inten.	Remarks
128	$2^h53^m47^s$	$+0°16'$	19″x 8″	16°	F	
129	3 09 31	+14 35	27 x 5	142	F	
130	3 09 49	+14 52	11 x 8	35	F	
131	3 10 31	+15 12	27 x 8	154	F	
132	3 10 43	+14 43	14 x14	...	F	
133	3 10 54	+15 34	11 x 8	0	F	
134	3 11 01	+15 16	11 x 5	163	B	Stellar.
135	3 11 11	+15 16	14 x14	...	MB	
136	3 11 51	+14 55	16 x11	50	F	
137	3 12 04	+14 45	14 x 5	35	MB	
138	3 12 04	+15 10	35 x11	45	MB	Bright nucleus.
139	3 12 09	+14 34	11 x 8	0	MF	Stellar.
140	3 12 19	+14 30	14 x 8	63	F	
141	3 12 26	+15 39	14 x11	0	F	
142	3 49 11	− 0 24	11 x11		F	
143	3 50 51	− 0 26	14 x 5	143	MF	
144	3 51 51	− 0 19	14 x 8	58	MF	
145	4 01 49	+74 46	8 x 5	57	MF	Stellar.
146	4 07 54	+75 20	8 x 5	0	F	
147	4 11 06	−15 32	19 x 8	108	F	Star at western end.
148	4 11 08	−14 34	14 x 8	148	MF	
149	4 11 22	−15 30	11 x 8	0	MF	
150	4 11 33	−14 52			MB	Stellar.
151	4 12 54	−15 29			MF	Stellar.
152	4 13 58	−15 39	8 x 5	156	F	
153	4 14 16	−15 01	19 x11	165	MF	
154	4 14 20	−15 04	5 x 5	...	MB	
155	4 14 26	−15 13	1′35″x 8″	21	F	MF nucleus.
156	4 14 36	−14 42	14 x 8	33	F	
157	4 15 08	−14 43	14 x14		F	
158	4 15 10	−15 40	14 x14		F	
159	4 15 18	−14 38	16 x11	90	F	
160	4 21 11	+29 44	27 x11	163	VF	
161	4 23 03	+30 23	22 x 5	42	F	
162	4 45 52	− 0 32	27 x16	65	F	Stellar.
163	4 46 10	− 0 30	14 x 8	116	MB	
164	4 46 38	+ 0 16	27 x14	110	F	
165	4 47 18	+ 0 08	8 x 8		MF	
166	4 47 38	− 0 01	11 x 5	153	F	
167	4 47 52	+ 0 23	8 x 5		MB	Stellar.
168	4 47 56	+ 0 07	41 x 5	140	MB	Nucleus at center
169	4 48 04	+ 0 22	33 x 5	90	F	Nucleus at center
170	4 48 18	+ 0 26	16 x 3	109	F	
171	4 48 22	− 0 07	8 x 5	11	F	
172	4 48 24	− 0 05	8 x 5	0	F	
173	4 49 16	+ 0 08	14 x11	0	F	
174	4 49 40	− 0 02	41 x11	94	F	
175	4 50 04	+ 0 05	14 x14	...	F	
176	4 50 18	+ 0 03	14 x 8	160	F	
177	4 50 24	− 0 13	24 x14	142	MB	
178	5 14 52	−14 49	5 x 5		F	Stellar.
179	5 14 54	−14 50	3 x 3		F	Stellar.
180	5 15 28	−14 44	5 x 5		MF	Stellar.
181	5 16 43	−14 46	14 x 8	150	F	
182	5 37 45	+45 00	14 x14	152	B	Spiral.
183	5 56 52	− 0 29	14 x 5	84	MB	
184	7 03 39	+59 51	54 x11	20	VF	
185	7 09 31	+59 45	8 x 8	...	VF	
186	7 10 21	+59 50	8 x 8	...	F	
187	7 11 39	+59 31	54 x24	158	F	Bright nucleus.
188	7 16 27	+15 25	8 x 5	...	MB	Stellar.
189	7 16 55	+15 09	19 x 5	48	F	MB nucleus.
190	$7^h17^m16^s$	$+15°02'$	19″x 8″	102°	F	
191	7 35 05	+45 16	11 x11	...	F	
192	7 36 07	+44 27	14 x11	63	F	
193	7 36 37	+45 20	8 x 8	...	F	Stellar.
194	7 36 42	+44 45	11 x 5	42	VF	
195	7 37 28	+44 56	16 x 5	163	MB	
196	7 37 43	+44 47	3 x 3	...	MB	Stellar.
197	7 37 53	+44 31	11 x11	...	MF	
198	7 37 57	+44 39	8 x 8	...	MF	
199	7 38 01	+44 44	16 x11	125	F	Bright nucleus.
200	7 38 15	+44 21	19 x16	?	MF	
201	7 38 47	+44 37	33 x 5	87	MB	
202	7 39 37	+45 05	16 x11	115	MF	
203	7 40 08	+44 28	8 x 5	168	F	
204	7 40 13	+45 00	35 x14	175	F	Bright nucleus.
205	7 40 31	+44 29	27 x11	164	MB	
206	7 41 15	+44 41	22 x 8	132	MB	
207	7 42 11	+44 54	14 x11	?	MB	Stellar.
208	7 42 35	+44 35	11 x 8	54	F	
209	7 48 45	− 0 14	14 x14	...	MF	
210	8 11 22	+74 16	33 x11	135	MF	
211	8 14 27	+74 42	19 x 8	51	F	
212	8 15 17	+15 16	8 x 8		F	
213	8 15 17	+14 46	11 x 8	8	F	
214	8 15 37	+14 48	11 x 5	77	MB	
215	8 18 51	+15 02	22 x16	0	B	
216	8 23 42	+30 29	16 x 8	90	F	
217	8 23 55	+30 34	19 x19		F	
218	8 24 00	+30 16	14 x11	90	F	
219	8 24 05	+30 17	8 x 5	0	F	
220	8 24 17	+29 31	22 x11	38	F	
221	8 24 18	+30 16	14 x 8	112	F	
222	8 24 35	+29 35	22 x11	12	MF	
223	8 24 38	+29 36	19 x11	10	F	
224	8 25 17	+30 10	8 x 5	111	F	
225	8 25 38	+30 05	5 x 5	...	MB	Stellar.
226	8 26 04	+29 51	14 x11	119	F	
227	8 26 08	+29 58	5 x 5	...	F	
228	8 26 35	+29 32	19 x 8	10	MB	
229	8 26 46	+30 00	8 x 5	?	B	Nucleus at one end.
230	8 26 55	+29 57	27 x22	...	F	Irregular.
231	8 27 02	+30 14	41 x11	12	B	
232	8 27 10	+29 31	27 x11	166	MF	
233	8 27 48	+30 09	14 x 8	13	F	
234	8 27 49	+29 45	14 x11	176	MF	
235	8 27 58	+30 31	14 x14		MB	
236	8 28 03	+29 55	14 x11	25	F	
237	8 28 05	+29 48	46 x27	63	MF	
238	8 28 20	+29 51	16 x 5		F	
239	8 28 27	+29 45	14 x14	...	F	
240	8 28 42	+30 33	16 x11	...	MF	Near corner of plate.
241	8 29 02	+30 13	22 x 8	155	MF	
242	8 29 03	+30 13	8 x 8	...	F	Stellar.
243	8 29 38	+30 01	41 x27		MF	Irregular.
244	8 38 37	+44 42	22 x11	39	F	
245	8 39 00	+44 56	14 x14	...	MF	
246	8 39 14	+44 51	5 x 5	...	B	Stellar.
247	8 39 14	+44 52	19 x 5	99	MF	
248	8 39 15	+44 51	19 x 5	150	MB	
249	8 39 52	+45 06	14 x 3	171	F	
250	8 41 24	+44 31	19 x 5	31	F	

No	α	δ	LxW	Elong.	Inten	Remarks
251	8h11m40s	+14°55′	22″x11″	169°	MB	
252	8 11 54	+41 53	5 x 5		MF	
253	9 01 31	−59 26	24 x14	1	MF	
254	9 09 13	+14 38	11 x11		F	
255	9 09 58	+14 24	8 x 8		MB	Stellar.
256	9 10 11	+14 05	14 x 8	0	F	
257	9 10 43	+14 56	14 x 8	155	F	
258	9 10 51	+14 07	11 x 5	60	F	
259	9 10 53	+13 57	27 x11	165	F	
260	9 10 53	+14 12	5 x 3	61	F	
261	9 10 59	+13 59	14 x14		F	
262	9 11 23	+15 08	5 x 5		F	
263	9 11 33	14 59	5 x 3	90	F	Stellar.
264	9 11 34	−14 58	8 x 5	130	F	Stellar.
265	9 11 38	−14 41	8 x 8		F	Stellar.
266	9 12 01	−15 30	8 x 5	0	F	
267	9 12 11	15 29	14 x 5	155	F	
268	9 12 32	−15 38	19 x11	106	F	Double.
269	9 12 33	+14 11	14 x 5	75	F	
270	9 12 56	−15 31	8 x 5	150	F	
271	9 13 00	−14 57	14 x 8	48	F	
272	9 13 01	−15 21	8 x 5	151	F	
273	9 13 04	−15 23		.	F	Stellar.
274	9 13 09	+13 51	27 x16	130	MF	MB nucleus.
275	9 13 53	−15 27	22 x 8	166	MF	
276	9 13 58	−15 33	14 x 8	163	MF	
277	9 21 19	+29 47	27 x14	145	MB	Nucleus at center.
278	9 22 21	+29 56	14 x 5	15	MB	
279	9 22 27	+29 48	16 x16	.	F	
280	9 22 51	+30 01	8 x 5	47	MB	
281	9 23 13	+29 55	5 x 5	.	F	
282	9 23 41	+30 28	19 x 3	10	F	
283	9 23 45	+29 51	16 x11	0	F	
284	9 23 49	+30 31	22 x11	147	MB	Star involved.
285	9 24 41	+30 31	8 x 5	0	F	
286	9 24 43	30 23	14 x11	30	MB	
287	9 25 51	+29 51	19 x 5	10	MF	Faint nucleus at center.
288	9 25 55	+30 25	54 x14	11	MB	Faint nucleus at center.
289	9 26 19	+29 19	5 x 3	90	F	
290	9 26 19	+30 22	11 x11		F	
291	9 26 2	+29 32	8 x 8		MF	
292	9 26 34	+30 11	41 x14	45	F	Bright nucleus.
293	9 27 46	+29 33	5 x 5		F	
294	9 36 04	44 35	19 x11	115	F	
295	9 38 01	+44 43	11 x 3	39	F	
296	9 41 13	+44 23	27 x14	45	F	
297	9 49 35	− 0 22	16 x11	60	F	
298	9 49 48	0 18	14 x 8	162	F	
299	9 50 21	− 0 20	8 x 8		MF	Stellar.
300	9 50 53	− 0 08	8 x 5		F	Stellar.
301	9 51 13	− 0 20	5 x 5		MB	Stellar.
302	9 51 19	− 0 14	22 x 5	135	MF	
303	9 51 39	− 0 32	27 x 5	0	F	
304	9 51 41	− 0 07	8 x 5	10	F	
305	9 51 51	− 0 12	11 x 8	19	F	
306	9 52 18	− 0 07	5 x 5		F	
307	9 52 25	+ 0 09	5 x 3	151	MF	
308	9 52 35	+ 0 03	11 x11		F	
309	9 52 35	0 00	14 x 5	106	F	Small nucleus.
310	9 52 43	− 0 02	8 x 5	21	MF	

No.	α	δ	LxW	Elong.	Inten	Remarks
311	9h53m05s	+ 0°25′	41″x 8″	41°	F	
312	9 53 06	+ 0 07	11 x 8	71	MF	
313	9 53 11	+ 0 08	5 x 5	.	MF	Stellar.
314	9 53 11	0 00	8 x 5	106	MF	
315	9 53 13	+ 0 10	8 x 8		MB	Stellar.
316	9 53 38	− 0 15	14 x11	129	F	
317	9 53 46	− 0 25	8 x 8		F	
318	9 53 51	0 00	19 x 8	150	MF	
319	9 53 51	− 0 26	11 x11		F	
320	9 53 53	− 0 25	8 x 8		F	
321	9 53 59	− 0 25	8 x 8		F	
322	9 53 59	+ 0 30	19 x14	29	F	
323	9 54 01	− 0 30	11 x 8		MF	Stellar.
324	9 54 33	+ 0 26	22 x11	49	F	Stellar.
325	9 54 53	+ 0 10	22 x16	.	MB	Stellar.
326	10 12 08	+15 28	16 x16		F	
327	10 13 10	+15 34	16 x 5	145	F	
328	10 13 27	−15 01	.		MF	Stellar.
329	10 13 41	+14 40	14 x 5	0	MB	
330	10 14 35	−14 52	14 x14	.	MF	
331	10 14 38	+14 50	27 x11	152	MF	
332	10 14 41	−15 22	.		MF	Stellar.
333	10 15 05	−14 40	11 x 8	131	F	
334	10 15 39	14 53	19 x 8	14	MF	
335	10 15 42	+14 45	16 x 8	160	MF	
336	10 15 51	−15 00	8 x 5	33	MF	
337	10 16 24	+15 11	27 x11	29	F	
338	10 16 49	−15 09	16 x 5	150	F	
339	10 16 54	+15 21	14 x14		F	
340	10 17 07	−14 51	.		MF	Stellar.
341	10 17 21	−14 49	14 x14		MF	
342	10 18 09	14 49	11 x11	55	F	
343	10 22 38	+30 19	8 x 5		MF	Stellar.
344	10 23 34	+29 36	8 x 8		MB	
345	10 34 56	+45 04	11 x 8	57	F	
346	10 35 26	+45 38	14 x14		F	
247	10 37 45	+45 00	14 x 3	98	VF	
348	10 37 46	+44 59	5 x 3	17	B	
349	10 39 11	+44 41	5 x 5		F	
350	10 41 02	+45 14	5 x 5		F	
351	10 49 16	− 0 39	11 x 5	0	F	
352	10 49 17	− 0 20	5 x 5		VF	
353	10 49 19	0 20	11 x 5	143	VF	
354	10 49 51	− 0 40	8 x 8	.	F	
355	10 50 03	− 0 44	8 x 5	127	F	
356	10 50 25	+ 0 06	8 x 8	.	VF	
357	10 50 25	− 0 40	11 x11		VF	
358	10 50 29	− 0 37	8 x 5	0	F	
359	10 50 39	− 0 27	22 x 5	60	F	
360	10 50 42	0 05	16 x 3	101	F	
361	10 50 45	− 0 28	11 x 5	115	F	
362	10 51 25	− 0 29	3 x 3		MF	
363	10 51 47	− 0 25	5 x 5		MF	
364	10 51 59	0 31	11 x 8	62	VF	
365	10 52 01	− 0 35	16 x 8	108	VF	
366	10 53 35	− 0 18	33 x11	115	MF	
367	10 53 37	0 32	11 x 8	90	B	
368	10 53 38	− 0 35	14 x 8	53	B	
369	10 59 38	+60 11	14 x19	90	MB	
370	10 59 42	+60 10	16 x11	77	MF	
371	10 59 54	+60 15	8 x 3	21	MF	
372	10 59 56	+60 09	16 x14	80	F	

No.	α	δ	LxW	Elong.	Inten.	Remarks
373	$11^h00^m34^s$	+59°26′	. .	.	MB	
374	11 01 19	+59 41	8″x 5″	90°	F	
375	11 03 19	+59 15	22 x 8	75	MF	
376	11 04 00	+59 43	5 x 3	0	F	
377	11 04 17	+59 45	8 x 5	176	F	
378	11 04 36	+59 42	19 x 5	33	MF	
379	11 05 14	+59 15	22 x22	.	F	
380	11 05 19	+59 26	8 x 5	155	F	
381	11 05 56	+59 59	16 x 8	24	MF	Star at S. end.
382	11 09 14	+59 36	27 x11	130	F	
383	11 11 02	−15 18	16 x14	90	MF	
384	11 11 22	−15 39	22 x 8	24	F	
385	11 11 36	−14 56	27 x16	145	F	
386	11 12 15	−15 31	14 x11	173	F	
387	11 14 45	−15 15	16 x 8	90	MB	Stellar.
388	11 15 30	−15 15	11 x 8	101	F	
389	11 15 39	+15 01	8 x 8	.	F	
390	11 16 10	+14 38	11 x 8	58	F	
391	11 16 23	+14 22	8 x 8	.	F	
392	11 16 23	+14 24	8 x 8	.	VF	
393	11 17 07	+14 29	22 x 8	170	MF	
394	11 17 24	+15 13	16 x 8	36	F	
395	11 17 57	+14 47	11 x 5	41	F	
396	11 18 21	+15 03	8 x 5	24	F	
397	11 19 17	+14 45	8 x 5	12	VF	
398	11 19 51	+14 33	22 x14	150	F	
399	11 19 55	+14 54	11 x 5	68	F	
400	11 19 57	+14 58	5 x 5		F	
401	11 21 11	+15 09	22 x11	21	F	
402	11 21 19	+14 29	14 x14		F	
403	11 28 42	+29 26	14 x14		F	
404	11 29 12	+29 29	14 x14	. . .	F	
405	11 30 23	+30 23	. .		F	Stellar.
406	11 30 26	+30 03	14 x 8	26	F	
407	11 32 31	+30 40	19 x11	15	F	
408	11 34 02	+44 22	5 x 5		F	
409	11 34 06	+44 49	22 x 8	118	MF	
410	11 36 14	+44 59	27 x11	79	F	
411	11 36 42	+45 00	5 x 5	. . .	F	
412	11 48 12	+ 0 07	14 x11	15	F	
413	11 49 16	− 0 01	14 x 8	28	F	
414	11 49 34	− 0 38	16 x11	95	MF	
415	11 49 36	+ 0 01	14 x11	17	F	
416	11 50 16	− 0 28	19 x11	2	F	
417	11 50 52	− 0 04	11 x11	. .	F	
418	11 51 11	− 0 03	11 x 5	42	F	
419	11 51 22	− 0 05	11 x 5	90	F	
420	11 51 24	+ 0 24	16 x14	18	F	
421	11 51 34	− 0 03	8 x 8	.	F	
422	11 51 34	+ 0 17	8 x 5	0	F	
423	11 51 38	+ 0 18	16 x11	150	F	
424	11 51 44	+ 0 18	11 x11	.	F	
425	11 52 02	− 0 01	8 x 8	.	F	
426	11 52 19	− 0 28	19 x 8	80	F	
427	11 55 13	+29 39	22 x14	57	F	
428	11 55 25	+29 56	14 x11		F	Near edge of
429	11 55 53	+29 39	14 x 8	90	F	[plate.
430	11 55 57	+29 29	16 x14	?	F	
431	11 56 12	+30 23	8 x 8	. . .	VF	
432	11 56 24	+29 05	19 x11	15	F	
433	11 56 33	+29 51	8 x 8	. . .	F	
434	11 56 35	+29 47	14 x 5	?	MF	
435	$11^h57^m02^s$	+29°28′	14″x 5″	90°	F	
436	11 57 04	+29 24	19 x11	63	F	
437	11 57 08	+29 10	14 x11	0	F	
438	11 57 11	+29 52	16 x11	70	F	
439	11 57 12	+29 05	14 x11	0	F	
440	11 57 28	+29 21	8 x 8		B	
441	11 57 35	+29 21	5 x 5	.	F	
442	11 57 41	+30 00	19 x 8	56	F	
443	11 57 48	+29 35	8 x 5	141	MB	
444	11 57 54	+29 32	8 x 5	90	MB	
445	11 57 58	+29 59	19 x11	130	MF	
446	11 58 02	+29 31	8 x 5	0	MF	
447	11 58 07	+29 36	5 x 3	11	F	
448	11 58 14	+29 57	8 x 8		F	
449	11 58 18	+29 12	14 x14		F	
450	11 58 18	+29 34	11 x 5	111	MF	
451	11 58 22	+29 07	14 x 8	18	F	
452	11 58 24	+29 06	11 x11		F	
453	11 58 24	+29 35	5 x 5		F	
454	11 58 24	+29 45	8 x 8		F	
455	11 58 36	+29 46	5 x 5		F	
456	11 58 38	+29 58	8 x 5	37	F	
457	11 58 39	+29 46	5 x 5	.	F	
458	11 58 40	+29 37	5 x 5		F	
459	11 58 47	+30 02	8 x11	171	F	
460	11 58 56	+29 28	8 x 5	164	F	
461	11 58 57	+29 16	8 x 8	. . .	MF	
462	11 59 00	+29 11	8 x 8	.	F	
463	11 59 04	+30 07	24 x11	43	MF	
464	11 59 17	+29 35	3 x 3	. .	F	
465	11 59 22	+29 25	11 x 3	53	F	
466	11 59 24	+29 40	8 x 5	49	F	
467	11 59 31	+29 27	5 x 5	.	F	
468	11 59 38	+29 25	8 x 8	. .	VF	
469	11 59 43	+29 12	11 x 5	120	F	
470	11 59 50	+29 45	3 x 3	. .	F	
471	11 59 51	+29 49	11 x 8	52	F	
472	11 59 52	+29 20	8 x 8	. .	MF	
473	11 59 53	+29 49	11 x 5	103	F	
474	11 59 54	+29 25	8 x 5	90	F	
475	12 00 00	+30 09	5 x 5		F	
476	12 00 05	+29 50	8 x 8		VF	
477	12 00 36	+30 07	8 x 8		F	
478	12 00 54	+29 41	11 x 8	97	B	
479	12 00 55	+29 31	11 x11	. .	F	
480	12 01 16	+29 06	27 x11	134	F	
481	12 01 20	+30 00	16 x 5	7	F	
482	12 01 23	+29 41	14 x11	20	F	
483	12 01 32	+29 53	8 x 8		VF	
484	12 10 07	+15 25	19 x14	19	F	
485	12 10 10	+14 29	14 x11	90	F	
486	12 10 11	+14 36	33 x11	144	F	
487	12 10 38	+14 49	22 x22	. . .	F	
488	12 10 53	+14 53	14 x14	. .	F	
489	12 10 55	+15 11	14 x 8	129	MB	
490	12 10 58	+14 26	16 x11	47	F	
491	12 11 08	+14 55	22 x14	51	F	
492	12 11 16	+15 21	19 x 8	162	MF	
493	12 11 18	+14 32	35 x16	30	F	
494	12 11 21	+14 42	14 x11	29	F	
495	12 11 21	+14 30	19 x11	155	F	
496	12 11 21	+14 55	16 x 8	11	F	

No.	α	δ	LxW	Elong.	Inten.	Remarks
497	12h 11m 21s	+14 57′	54″x22″	0°	MB	
498	12 11 22	+14 45	11 x 8	28	F	
499	12 11 25	+14 44	14 x 8	107	MF	
500	12 11 25	+14 31	24 x11	115	F	
501	12 11 27	+14 41	16 x11	34	B	
502	12 11 35	+14 30	41 x 5	15	MF	
503	12 11 40	+14 45	14 x11	62	F	
504	12 11 43	+14 13	11 x11		MF	
505	12 11 43	+14 39	19 x11	50	F	
506	12 11 47	+14 35	8 x 8		F	
507	12 11 51	+15 00	8 x 8		B	
508	12 11 55	+14 32	19 x11	135	MF	
509	12 12 01	+14 25	8 x 8		MF	Near edge of [plate.
510	12 12 00	−15 30	22 x16	47	F	
511	12 12 02	−15 25	19 x 8	54	F	
512	12 12 05	+15 02	16 x 8	140	MB	
513	12 12 05	+14 33	11 x 8	10	F	
514	12 12 07	+14 50	14 x 8	155	F	
515	12 12 29	+14 40	11 x 8	141	F	
516	12 12 39	+15 04	16 x11	80	F	
517	12 12 46	+14 55	8 x 5	44	F	
518	12 12 48	+15 10	5 x 5		F	
519	12 12 56	+15 23	19 x11	16	F	
520	12 12 59	+14 25	11 x14		F	
521	12 13 01	+14 27	16 x 8	162	F	
522	12 13 08	+15 14	16 x16		F	
523	12 13 09	+14 28	11 x11	160	F	
524	12 13 29	+14 36	5 x 5		F	
525	12 13 31	+15 15	14 x14		MF	
526	12 13 37	+14 27	8 x 5	164	F	
527	12 13 59	+15 19	11 x 8	11	F	
528	12 14 11	+15 21	5 x 5		F	
529	12 14 28	+14 29	16 x11	118	F	
530	12 14 29	−15 39	16 x16		F	
531	12 14 39	+14 23	11 x11		MB	
532	12 14 51	+14 59	5 x 5		F	
533	12 14 55	+14 29	27 x27		VF	
534	12 15 35	15 02	8 x 5	21	MF	
535	12 15 46	−15 00	8 x 5	81	F	
536	12 18 18	+75 20	8 x 8		MB	Stellar.
537	12 36 35	− 0 15	8 x 8		F	
538	12 36 59	+ 0 22			MF	Stellar.
539	12 37 11	− 0 22	11 x 8	58	F	
540	12 37 23	− 0 11			F	Stellar.
541	12 38 19	− 0 11	1′48 x27	70	F	
542	12 39 53	+ 0 02			F	Stellar.
543	12 40 21	+ 0 27		..	MF	Stellar.
544	12 47 43	+14 35	27 x19	64	F	
545	12 48 16	+14 21	5 x 5		F	Stellar.
546	12 50 00	+14 31	14 x11	157	F	
547	12 50 02	+14 25	8 x 8		F	
548	12 50 08	+15 30	27 x 5	175	F	
549	12 50 15	+14 25	11 x11		F	
550	12 50 28	+14 21			MB	Stellar.
551	12 50 40	+14 39	41 x19	106	F	Stellar nucleus.
552	12 51 17	+14 25	33 x14	148	F	
553	12 51 23	+14 47	22 x 8	143	MF	
554	12 51 30	+14 44	8 x 5	115	F	
555	12 51 38	+14 29	19 x14	104	F	
556	12 51 41	+14 29	22 x11	108	F	
557	12 52 23	+15 10	27 x11	35	F	
558	12 53 00	+15 00	14 x11	20	MF	
559	12h 53m 08s	+15° 05′	14″x11″	...	MF	
560	13 01 57	+29 33	16 x14	98°	F	
561	13 02 14	+29 23	11 x11		VF	
562	13 03 22	+29 19	14 x 5	6	F	
563	13 03 47	+29 29	22 x 8	0	F	
564	13 04 00	+29 30	14 x 8	156	F	
565	13 04 21	+29 22	11 x 8	34	F	
566	13 04 29	+29 21	19 x 8	0	F	
567	13 04 34	+29 22	22 x11	3	MF	
568	13 05 42	+29 18	14 x14	...	VF	
569	13 06 30	+29 28	19 x 8	101	F	
570	13 10 02	+14 39	33 x16	82	F	
571	13 11 07	+15 02	14 x14	.	F	
572	13 11 38	+14 36	11 x11		F	
573	13 11 55	+14 36	11 x11	.	F	
574	13 12 14	+15 04	11 x 5	35	VF	
575	13 12 21	+14 54	41 x19	0	MB	2 branched spiral. [N. br. shorter [and more curved [than S.
576	13 12 25	+14 53	14 x 5	120	MB	
577	13 12 57	−15 03	8 x 5	90	MF	
578	13 12 58	−15 01	8 x 5	90	MF	
579	13 13 12	+14 44	8 x 3	153	F	
580	13 13 14	+14 24	16 x 5	104	VF	
581	13 13 14	−14 50	27 x 8	178	F	
582	13 13 16	−14 50	30 x11	147	F	
583	13 13 26	−15 13	41 x24	154	MF	
584	13 13 34	+15 10	19 x 8	35	F	
585	13 13 36	+14 55	8 x 5	79	F	
586	13 13 36	−15 18	41 x19	34	MB	
587	13 13 48	+14 15	5 x 5		MF	
588	13 13 54	−14 35	41 x11	19	F	
589	13 13 59	+14 32	14 x 5	72	F	
590	13 14 00	+14 21	14 x14		F	
591	13 14 09	+14 31	14 x 8	123	MB	Spiral (?)
592	13 14 22	−15 33	27 x14	0	VF	
593	13 14 28	−15 27	16 x11	17	F	
594	13 14 30	−15 21	8 x 5	90	F	
595	13 14 30	−14 10	41 x14	13	MF	2 nuclei.
596	13 14 37	−14 40	11 x11	.	MF	
597	13 14 41	−15 23	8 x 5	69	F	
598	13 15 04	+15 02	16 x11	172	MB	
599	13 15 12	−14 45	8 x 8		F	
600	13 15 27	+14 51	16 x14	57	MB	
601	13 16 23	−15 21	11 x 8	7	F	
602	13 16 27	−15 21	16 x11	160	F	
603	13 16 28	−15 26	11 x 8		MB	Stellar. Near [corner of plate.
604	13 16 38	−15 38	16 x 8	150	F	
605	13 16 48	−15 01	41 x11	48	F	
606	13 16 51	−15 29	16 x11	122	MF	
607	13 17 38	+59 01	11 x11		MF	
608	13 20 30	+59 45	8 x 8		MF	
609	13 20 31	+58 53	5 x 3	14	F	
610	13 20 37	+59 15	16 x 3	108	MF	
611	13 20 58	+59 30	5 x 5	...	VF	
612	13 21 14	+59 48	19 x 8	77	MB	
613	13 21 18	+58 58	11 x 5	21	F	
614	13 21 20	+58 57	19 x 5	173	VF	
615	13 21 42	+58 57	8 x 5	.	F	Stellar.
616	13 22 34	+59 33	3 x 3	.	F	Stellar.
617	13 23 18	+59 37	11 x 8	172	MF	
618	13 24 04	+59 53	19 x11	13	F	
619	13 24 26	+58 54	8 x 3	0	VF	

No.	α	δ	LxW	Elong.	Inten.	Remarks
620	13h24m29s	+58°54′	8″x 3″	170°	F	
621	13 25 06	+59 33	14 x14		F	
622	13 25 39	+59 29	5 x 5		VF	
623	13 25 40	+59 45	14 x 8	13	F	
624	13 25 56	+59 30	8 x 8		VF	V. close to b. star.
625	13 25 56	+59 31	14 x 3	50	VF	V. close to b. star.
626	13 26 04	+59 29	5 x 5		F	Stellar.
627	13 26 12	+58 53	1′35 x41	0	MF	Many stellar [nuclei. Irreg. [with a number [of condensations.
628	13 27 08	+58 48	5 x 5		F	Stellar.
629	13 27 34	+59 20	5 x 5		VF	
630	13 27 39	+59 23	5 x 5		F	Stellar.
631	13 33 50	− 0 40	14 x 8	155	MB	
632	13 34 18	− 0 21		.	F	Stellar.
633	13 35 15	− 0 45	14 x11	148	F	
634	13 35 15	− 0 10	14 x 8	70	F	
635	13 46 54	+45 17	19 x11	136	F	
636	13 49 46	+45 30	5 x 5	. .	F	
637	13 50 30	+44 39	14 x 8	10	F	
638	13 53 19	+44 55	27 x 8	170	B	Shaped like V.
639	13 56 57	+30 00	19 x 5	78	F	
640	13 57 01	+29 58	27 x27		F	
641	13 58 29	+29 49	5 x 3	90	MF	
642	13 58 30	+29 09	8 x 8		F	
643	13 59 43	+29 48	19 x 5	176	F	Star involved.
644	14 00 12	+30 06	5 x 5	. .	F	
645	14 00 33	+29 36	19 x 5	161	MF	
646	14 00 35	+29 38	11 x11		F	
647	14 00 37	+29 01	14 x 8	12	MF	
648	14 01 11	+29 51	16 x 8	170	F	
649	14 01 14	+29 52	19 x 5	13	F	
650	14 01 35	+29 51	5 x 5		F	
651	14 01 37	+29 32	5 x 5		F	
652	14 02 00	+29 38	8 x 5	90	F	
653	14 02 05	+29 20	8 x 8		F	
654	14 02 13	+30 06	8 x 5	0	F	
655	14 03 51	+29 19	16 x14	0	F	
656	14 08 40	−14 53	11 x 8	48	F	
657	14 13 08	+14 52	14 x 8	112	F	
658	14 13 08	+15 01	8 x 5	70	VF	
659	14 14 04	+15 03	8 x 5	74	F	
660	14 14 11	+15 35	14 x14	. .	F	
661	14 14 15	+15 25	19 x 5	102	F	
662	14 14 36	+15 35	8 x 8	. . .	VF	
663	14 15 18	+15 41	14 x11	90	F	
664	14 15 26	+15 21	19 x 8	41	F	
665	14 15 30	+15 11	11 x 5	39	MF	
666	14 15 42	+15 04	14 x11	90	F	
667	14 15 43	+15 29	5 x 5		F	
668	14 15 54	+15 05	14 x11	90	F	
669	14 16 01	+15 45	8 x 5	11	F	
670	14 16 18	+15 51	19 x19	. .	F	
671	14 16 35	+15 24	33 x16	175	MF	
672	14 18 01	+15 19	14 x 8	90	MF	
673	14 18 11	+15 29	27 x14	165	MB	
674	14 45 04	+44 48	35 x14	123	F	
675	14 45 24	+44 49	33 x 8	140	MF	
676	14 46 14	+45 14	8 x 8	. .	MF	
677	14 47 49	+45 25	27 x14	169	F	
678	14 48 24	+44 53	16 x11	108	F	
679	14h50m02s	+45°33′	27″x16″	74°	MF	
680	15 00 01	+29 39	22 x14	45	F	
681	15 00 07	+29 16	8 x 8	.	F	
682	15 01 01	+29 40	22 x 8	63	F	
683	15 01 25	+29 07	11 x 8	62	F	
684	15 01 47	+29 37	14 x 8	140	MF	
685	15 02 13	+29 36	8 x 5	157	F	
686	15 02 23	+30 15	11 x11		F	
687	15 02 25	+29 45	14 x 8	34	MF	
688	15 02 41	+30 04	22 x 5	40	F	
689	15 02 51	+29 53	16 x16	.	F	
690	15 03 21	+29 49	5 x 3	90	F	
691	15 03 25	+29 49	5 x 5	.	VF	
692	15 03 33	+29 48	16 x 8	100	F	
693	15 03 57	+29 10	11 x11	.	F	
694	15 04 05	+29 03	14 x11	162	F	
695	15 04 07	+29 55	5 x 3	140	F	
696	15 04 15	+29 58	5 x 5	. .	VF	
697	15 04 18	+29 58	8 x 5	27	VF	
698	15 04 23	+29 58	8 x 5	50	VF	
699	15 04 26	+30 07	16 x 8	57	F	
700	15 04 33	+30 14	8 x 8		F	
701	15 04 33	+29 58	19 x 5	80	F	
702	15 04 35	+29 55	14 x14		F	
703	15 08 47	−15 07	2′16″x 2′16″		VF	Spiral.
704	15 09 16	+15 08	14 x11″	0	VF	
705	15 09 57	+14 13	19 x 8	15	F	
706	15 10 18	−15 20	11 x 8	72	F	
707	15 10 30	+14 41	19 x 5	130	F	
708	15 10 41	−15 37	11 x11	.	F	
709	15 11 29	+15 18	14 x11	120	VF	
710	15 12 18	−15 04	8 x 8		VF	
711	15 12 27	+59 11	41 x22	17	MF	
712	15 14 04	−15 40	11 x11		MF	Stellar.
713	15 14 49	+59 10	27 x14	13	F	
714	15 14 52	+59 45	.		F	Stellar.
715	15 15 35	+59 25	14 x11	0	F	Stellar.
716	15 16 47	+59 47	1′13″x 8	60	MF	
717	15 16 50	+59 18	19 x16	25	MF	
718	15 17 59	+59 27	8 x 5	54	F	
719	15 18 01	+59 45	19 x14	152	F	
720	15 18 15	+60 32	8 x 5	15	F	
721	15 18 27	+59 31	27 x14	101	F	
722	15 32 18	+ 0 12	8 x 5	66	MF	
723	15 33 46	− 0 10	8 x 3	163	MB	
724	15 34 18	+ 0 10	5 x 5	.	F	
725	15 36 06	+ 0 07	41 x14	40	F	
726	15 46 13	+44 59	14 x11	. .	MB	Stellar.
727	15 48 23	+45 09	16 x 8	145	F	
728	15 48 38	+44 30	8 x 5	3	MF	
729	15 50 06	+45 07	11 x 5	162	F	
730	15 51 56	+44 49	5 x 5	. . .	MF	Stellar.
731	15 56 06	+29 41	8 x 8	. . .	F	
732	15 56 34	+29 40	5 x 5	. . .	MF	
733	15 57 40	+30 05	27 x 5	137	F	
734	16 06 30	+74 51	5 x 5	. . .	F	
735	16 11 06	+15 03	14 x 8	145	F	
736	16 11 48	+15 31	16 x 8	130	MF	
737	16 11 50	+15 23	11 x11	. . .	F	
738	16 11 54	+15 28	14 x14	.	F	
739	16 11 56	+15 28	16 x 8	150	MF	
740	16 12 16	+14 44	5 x 5	. . .	F	

No.	α	δ	LxW	Elong.	Inten.	Remarks
741	16h12m20s	+15°06′	16″x 8	160°	MF	Double.
742	16 12 28	+14 50	14 x 8	108	F	
743	16 12 32	+14 52	14 x 3	142	F	
744	16 12 33	+14 56	19 x11	149	F	
745	16 12 35	+15 02	33 x 8	119	F	
746	16 12 39	+15 27	11 x11		F	
747	16 12 41	+14 17	3 x 3		MF	
748	16 12 44	+15 11	11 x 8	0	F	
749	16 12 45	+15 21	14 x 8	167	F	
750	16 13 00	+15 07	3 x 3		F	
751	16 13 10	+15 37	11 x11		F	
752	16 13 18	+15 08	11 x 5	171	F	
753	16 13 21	+15 22	14 x11	11	F	
754	16 13 30	+15 15	5 x 5		F	
755	16 14 20	+14 31	14 x14		F	
756	16 14 34	+14 46	11 x11		F	
757	16 17 02	+74 27	16 x11	0	F	
758	16 18 12	+75 21	27 x11	90	VF	
759	16 19 46	+74 35	14 x 5	0	F	
760	16 21 18	+75 02	8 x 8		F	
761	16 44 52	+45 18	5 x 5		F	
762	16 45 16	+45 57	14 x 8	156	MF	
763	16 46 10	+45 11	5 x 3	0	F	
764	16 48 08	+45 33	5 x 5		F	
765	16 48 14	+45 33	5 x 5		F	
766	16 55 51	+30 05	14 x14		MF	
767	16 56 22	+30 05	16 x16		F	
768	16 56 58	+29 37	19 x14	156	F	
769	16 57 38	+29 32	14 x11	90	F	
770	16 58 02	+30 27	11 x 8	130	F	
771	16 58 02	+30 18	11 x 8		MB	Stellar.
772	16 58 36	+29 18	5 x 5		F	
773	16 58 43	+29 10	5 x 5		MF	
774	16 59 34	+30 00	8 x 5	50	MF	
775	17 01 26	+29 26	14 x11	160	F	
776	17 02 12	+30 02	16 x11	114	VF	
777	17 24 21	+59 38	33 x11	109	F	
778	17 24 29	+59 32	54 x 8	90	F	
779	17 24 37	+59 12	19 x14	116	F	
780	17 24 42	+60 24	8 x 8		F	
781	17 25 44	+59 48	5 x 5	16	MF	
782	17 26 03	+60 06	14 x14		MF	
783	17 26 11	+59 13	5 x 5		B	
784	17 26 12	+59 12	22 x11	4	F	
785	17 26 15	+60 10	8 x 8		MF	
786	17 26 19	+60 03	19 x 8	11	MF	
787	17 27 11	+60 05	5 x 5	.	F	
788	17 28 15	+59 25	8 x 8		F	
789	17 28 53	+59 36	16 x16		F	
790	17 28 54	+60 25	14 x14		MB	Nucleus S end.
791	17 29 09	+59 42	33 x 8	33	MF	
792	17 29 13	+60 10	14 x14		F	
793	17 29 53	+60 27	8 x 5	90	F	
794	17 29 59	+60 02	8 x 8	.	F	
795	17 30 02	+59 34	41 x11	39	F	
796	17 30 18	+59 35	11 x 3	167	F	
797	17 30 49	+59 31	54 x 8	113	F	
798	17 31 07	+60 00	8 x 8		B	
799	17 31 14	+60 09	19 x 8	148	F	
800	17 31 49	+59 46	16 x 5	112	VF	
801	17 32 24	+59 33	11 x11	..	F	
802	17 46 36	+44 57	8 x 5	38	F	
803	17h47m26s	+45°01′	22″x11″	33°	F	
804	17 47 36	+45 25	14 x11	0	F	
805	17 48 04	+44 49	8 x 5	54	F	
806	17 48 18	+45 01	11 x 5	41	F	
807	17 49 03	+44 37	. .		MB	Stellar.
808	17 50 03	+45 00	. . .		MB	Stellar.
809	17 50 23	+45 00	16 x14	154	F	
810	17 50 53	+45 20	16 x11	90	F	
811	17 51 02	+45 26	16 x11	38	F	
812	19 20 50	+60 48	19 x 5	163	F	
813	19 24 35	+60 38	8 x 5	..	F	
814	19 41 12	+44 54	22 x19		MF	Somewhat irreg.
815	20 08 29	−14 45	14 x14		F	
816	20 08 58	−15 01	11 x 8	96	F	
817	20 09 00	−15 01	11 x 5	90	F	
818	20 09 46	−15 01	5 x 5	.	F	
819	20 11 11	−14 38	41 x19	0	F	F nucleus.
820	21 11 07	−14 45	.		MF	Stellar.
821	22 12 54	+15 46	. . . . . .	.	MF	Stellar.
822	22 13 03	+14 46	35 x 8	156	F	
823	22 14 08	+15 24	8 x 3	22	F	
824	22 35 51	− 0 11	. . . .	..	F	Stellar.
825	22 37 52	+ 0 43	14 x 8	53	F	
826	22 37 57	+ 0 35	5 x 3	45	F	
827	22 38 35	− 0 03	. . . .	. . .	F	Stellar.
828	22 38 38	− 0 03	.		F	Stellar.
829	22 38 39	− 0 04	..		F	Stellar.
830	22 38 43	− 0 01	8 x 5	128	F	
831	22 38 44	0 00	8 x 5	22	F	
832	22 58 46	+30 11	5 x 5		F	
833	23 01 33	+30 36	8 x 5	0?	B	Stellar.
834	23 03 38	+29 41	. .	..	F	
835	23 11 21	+14 59	16 x11	60	F	
836	23 11 23	+15 31	11 x 8	0	F	
837	23 11 53	+15 22	.		MF	Stellar.
838	23 11 55	+15 15			MF	Stellar.
839	23 12 07	+15 24	35 x11	36	MF	
840	23 12 16	+15 05	. . .		MB	Stellar.
841	23 12 41	+15 21	. .		MF	Stellar.
842	23 13 11	+15 38	.		MB	Stellar.
843	23 13 47	+14 49	5 x 3	9	F	
844	23 13 55	+14 57	8 x 5	111	B	
845	23 14 06	+14 48	19 x 8	43	MF	
846	23 14 09	+15 28	8 x 3	6	MF	
847	23 14 29	+15 22	16 x 5	141	F	
848	23 14 41	+15 13	8 x 8	.	F	Stellar.
849	23 15 02	+15 35	54 x14	81	F	
850	23 15 33	+15 27	1′00 x16	158	MF	
851	23 16 09	+15 42	27 x11	49	F	
852	23 16 27	+15 42	14 x11	90	MB	
853	23 36 27	+ 0 32	11 x11	. . .	VF	
854	23 39 10	+ 0 12	5 x 5	. . .	F	
855	23 39 24	+ 0 24	14 x 5	112	F	
856	23 39 24	+ 0 01	1′08 x 11	39	F	Bright nucleus.
857	23 39 33	+ 0 59	8 x 8	..	F	
858	23 39 38	+ 0 19	16 x11	140	MB	Possibly spiral.
859	23 39 55	+ 0 46	11 x 5	19	F	
860	23 40 20	+ 0 38	5 x 5	..	F	
861	23 40 26	+ 0 46	5 x 5	. . .	F	
862	23 40 28	+ 0 41	8 x 5	62	F	
863	23 40 32	+ 0 46	16 x 5	34	F	
864	23 41 46	+ 0 08	5 x 5	.	MF	

Table II is a list of previously discovered nebulæ which were found on the plates and which were measured in the same way as those of Table I. The following lists were available for comparison: Dreyer's New General Catalogue and the two supplementary Index Catalogues; Wolf's Königstuhl lists including No. XII; the Lick Observatory list found in Vol. VIII of the *Publications* of that observatory, and the second Harvard list in *Harvard Annals* 72, No. 2. In the column of identifications no distinction is made between the first and second Index Catalogues as the objects are numbered consecutively from the first to the second; the Wolf list and number and the Harvard number are given only if the object is not listed in the *N. G. C.* and *I. C.* The abbreviations are self-explanatory.

Table II.

No.	α	δ	LxW	Elong.	Inten.	Remarks
1	$0^h15^m12^s$	−14°38′	.	.	MB	IC 9 Stellar.
2	0 26 35	+30 18	5″x 5″		F	NGC 140 Stellar
3	1 28 47	+30 12	58′43 x36′00″		MF	NGC 598, M 33.
4	1 50 17	+ 0 22	8″x 8″		B	IC 172.
5	1 51 18	+ 0 50	27 x27		F	IC 173.
6	1 51 41	+ 0 53	14 x11	28°	F	IC 175.
7	2 14 44	+15 24	5 x 5		MB	NGC 882.
8	2 16 36	+15 21	5 x 5		MB	IC 1794.
9	4 12 04	+75 01	2′30 x 1′35	155	F	NGC 1530 Spiral [F except near [nucleus.
10	8 11 07	+74 15	14 x11″	?	B	NGC 2544.
11	8 14 02	+74 17	41 x16	100	MB	NGC 2550.
12	8 27 52	+29 50	1′08″x35	149	MB	NGC 2604. [Probably spiral.
13	9 22 33	+30 22	19 x19		B	IC 2476 Stellar.
14	9 22 35	+30 11	27 x11	115	B	IC 2475.
15	9 22 39	+30 24	8 x 8		F	IC 2478.
16	9 22 45	+30 23	19 x11	146	B	IC 2479.
17	9 22 59	+30 06	14 x11	. .	B	IC 2480 Stellar.
18	9 24 58	+29 55	27 x14	163	B	NGC 2893.
19	9 27 45	+30 19	16 x14		MB	IC 2490.
20	9 42 57	+44 30	54 x22	51	MF	NGC 2998. Al- [most certainly [spiral.
21	10 24 17	+29 57	4′32″x 1′8″	45	F	NGC 3254. Spir.
22	10 48 17	− 0 12	19″x11″	157	F	HN 1291.
23	10 48 45	− 0 07	35 x11	148	F	HN 1297.
24	10 48 45	− 0 12	16 x14	142	F	HN 1298.
25	10 49 47	+ 0 07	49 x 8	51	MF	IC 655.
26	11 17 19	+14 37	16 x 5	102	F	IC 2752.
27	11 17 19	+14 38	27 x 5	78	F	IC 2754.
28	11 17 35	+14 40	5 x 5		F	IC 2761.
29	11 17 41	+14 42	5 x 5		F	IC 2765
30	11 17 45	+14 41	8 x 8	.	MF	IC 2769.
31	11 19 45	+14 20	22 x11	161	F	IC 2799.
32	11 21 07	+15 10	41 x11	32	F	IC 2810.
33	11 34 44	+45 13	16 x11	18	F	W XII 76.
34	11 51 02	− 0 16	19 x 8	139	MB	HN 1481.
35	11 55 56	+29 30	11 x 8	?	MF	W VIII 360.
36	11 56 10	+29 26	11 x 5	18	F	W VIII 369.
37	$11^h56^m18^s$	+29°22′	19″x19″	.	MF	W VIII 375.
38	11 56 24	+29 32	14 x14	.	F	W VIII 378.
39	11 56 24	+30 12	14 x 8	0	F	W VIII 377.
40	11 56 24	+30 11	14 x 8	?	F	W VIII 376.
41	11 56 30	+29 29	19 x 8	139	MB	W VIII 381.
42	11 56 32	+30 11	8 x 8		F	W VIII 383.
43	11 56 57	+29 25	22 x14	63	MF	W VIII 404.
44	11 57 10	+29 55	16 x11	150	MF	W VIII 424.
45	11 57 10	+30 12	14 x 8	0	F	W VIII 415 or [417.
46	11 57 22	+30 01	16 x 5	146	MF	W VIII 425.
47	11 57 30	+29 35	8 x 3	98	F	W VIII 436.
48	11 57 50	+29 52	16 x11	60	MF	W VIII 453.
49	11 57 50	+29 58	41 x 8	61	B	W VIII 455.
50	11 57 54	+29 30	5 x 5		MB	W VIII 458.
51	11 57 57	+29 44	8 x 5	59	F	W VIII 460.
52	11 58 09	+29 21	14 x 5	108	MB	W VIII 465.
53	11 58 29	+29 57	5 x 5	.	MF	W VIII 473.
54	11 58 46	+29 55	1′35 x 8	58	MB	W VIII 483.
55	11 58 57	+30 12	16 x11	108	MF	W VIII 492.
56	11 58 58	+30 09	11 x11		F	W VIII 494.
57	11 59 12	+30 07	11 x11		F	W VIII 496.
58	11 59 16	+30 13	1′21 x 8	159	MF	W VIII 498.
59	11 59 29	+29 28	19 x16	171	MB	W VIII 512.
60	11 59 34	+29 25	19 x 8	90	F	W VIII 515.
61	11 59 34	+29 50	8 x 8	.	F	W VIII 516.
62	12 00 25	+29 21	19 x19		F	W VIII 557.
63	12 00 36	+29 16	27 x27		F	W VIII 561.
64	12 01 18	+29 56	14 x14		F	W VIII 586.
65	12 10 28	+14 32	2′02 x19	125	MB	IC 3061.
66	12 10 39	+14 56	27 x22	8	B	IC 3065.
67	12 10 50	+14 25	11 x11		MF	NGC 4208.
68	12 11 05	+14 25	2′02″x1′13″	66	MB	NGC 4212. Spiral. B nucleus
69	12 11 33	+14 54	14″x 8″	176	F	IC 3077.
70	12 11 39	+14 43	16 x14	131	MF	IC 3080.
71	12 12 08	+14 47	19 x11	0	B	IC 3093.
72	12 12 18	+15 01	54 x14	96	MB	IC 3096.
73	12 12 43	+14 42	8 x 5	0	MF	LO 364.
74	12 12 48	+15 07	5 x 5		F	LO 365.
75	12 12 51	+14 51	16 x 8	125	F	LO 366.
76	12 12 54	+15 09	8 x 8	.	F	LO 367.
77	12 12 55	+14 36	19 x 8	41	F	LO 368.
78	12 12 57	+14 59	5 x 5	.	F	LO 369.
79	12 13 14	+14 42	8 x 5	160	F	LO 371.
80	12 13 15	+14 35	22 x14	57	F	LO 372.
81	12 13 23	+14 41	14 x11	0	F	LO 375.
82	12 13 25	+14 51	5 x 5		F	LO 377.
83	12 13 29	+14 57	5 x 5		F	LO 378.
84	12 13 33	+15 00	8 x 5	40	F	LO 379.
85	12 13 38	+15 11	19 x14	120	F	LO 380.
86	12 13 39	+14 57	8 x 5	77	F	LO 381.
87	12 13 41	+15 10	14 x 8	30	MF	LO 382.
88	12 14 08	+15 01	8 x 8	.	F	LO 386.
89	12 14 09	+14 44	5 x 5	.	F	LO 387.
90	12 14 15	+14 44	5 x 5	.	F	LO 388.
91	12 14 17	+14 55	4′17 x 4′4″	45	MB	NGC 4254. Fine [spiral.
92	12 14 25	+15 01	5 x 3″	75	F	LO 390.
93	12 14 25	+15 03	11 x 5	64	F	LO 389.
94	12 14 29	+15 00	5 x 5	. . .	F	LO 391.
95	12 14 31	+14 29	22 x16	151	MF	IC 3142.

No	α	δ	LxW	Elong.	Inten.	Remarks
96	12h14m37s	+14 38	14″x14′		F	LO 393.
97	12 14 39	+15 02	16 x 8	60°	MF	LO 394.
98	12 14 57	+14 51	11 x 8	11	F	LO 397
99	12 14 58	+15 23	33 x30	?	B	NGC 4262.
100	12 15 03	+14 55	11 x11		F	LO 398.
101	12 15 13	+14 17	11 x11		F	LO 399.
102	12 15 31	+15 28	33 x16	?	MB	IC 781.
103	12 15 35	+14 38	16 x 8	90	F	LO 402.
104	12 15 41	+15 12	5 x 5		VF	LO 404 (?)
105	12 15 53	+15 03	11 x11		MF	LO 405.
106	12 16 03	+14 13	8 x 8		F	LO 406.
107	12 19 38	+75 27	27 x22	0	MF	NGC 4363.
108	12 32 13	+74 15	1′00 x 5	1	MF	IC 802.
109	12 33 57	+74 11	33 x22	100	VB	NGC 4589.
110	12 37 55	+ 0 25	2 16 x54	55	MF	NGC 4632 Spiral
111	12 38 23	+74 55	22 x16	80	VB	NGC 4648.
112	12 38 40	− 0 09	1 21 x27	31	MF	NGC 4642. Spiral
113	12 39 14	− 0 01	1 18 x1′35″	15	F	NGC 4653. Spiral
114	12 40 32	+ 0 02	1 32 x54″	36	MF	NGC 4666. Spiral
115	12 40 55	− 0 03	1 08 x27	0	MB	NGC 4668
116	13 01 10	+29 30	35 x 8	30	F	W IX 68.
117	13 01 12	+29 46	14 x14		B	W IX 69.
118	13 02 00	+29 33	46 x14	140	B	NGC 4966. Alm't cer. spiral.
119	13 02 04	+29 39	11 x 8	71	MF	W IX 80
120	13 02 04	+30 08	1′08″x14	90	F	W IX 81
121	13 02 16	+29 40	19 x11	0	MF	W IX 84.
122	13 02 17	+29 51	16 x11	0	MF	W IX 85.
123	13 02 20	+29 20	41 x19	0	MF	W IX 91. Somewhat irreg.
124	13 02 25	+29 51	5 x 5	.	MF	W IX 93.
125	13 02 30	+29 37	11 x 5	38	F	W IX 94
126	13 02 45	+30 23	11 x11		MF	W IX 97.
127	13 03 40	+29 11	14 x14		MB	W IX 107.
128	13 03 55	+29 20	27 x11	22	F	W IX 117.
129	13 04 38	+29 31	27 x19	90	MF	W IX 129.
130	13 04 38	+30 02	11 x 5	130	MB	W IX 130.
131	13 04 40	+29 13	41 x16	10	F	W IX 134
132	13 04 52	+29 22	11 x11		MB	W IX 135.
133	13 04 58	+29 51	14 x11	77	B	W IX 137. (?)
134	13 05 14	+29 28	14 x 5	90	MF	W IX 142.
135	13 05 30	+29 23	1 18 x1′8″	132	MF	NGC 5000. Spiral. Looks somewhat like letter S.
136	13 05 44	+30 31	24 x14″	153	MF	W IX 152 or 153
137	13 06 32	+30 11	41 x22	5	MF	IC 1210.
138	13 06 45	+30 07	16 x16		B	NGC 5004.
139	13 06 45	+30 03	38 x16	170	MF	W IX 162.
140	13 31 00	− 0 28			MB	IIN 1688.
141	13 31 33	− 0 35	24 x11	147	MF	IIN 1693.
142	13 31 35	− 0 03	19 x 8	143	MF	IIN 1694.
143	13 31 36	− 0 35	19 x11	0	MF	IIN 1695.
144	13 31 45	0 00	27 x14	152	MF	IIN 1696.
145	13 33 31	+ 0 29	14 x11	.	MB	IIN 1707.
146	13 33 50	+ 0 13	19 x11	21	B	IC 903.
147	13 34 10	+ 0 01	27 x11	155	MF	IIN 1709.
148	13 34 36	− 0 40	16 x11	90	MF	IIN 1713.
149	13 35 14	+ 0 07	8 x 5	0	MB	IIN 1711.
150	14 35 14	− 0 20	41 x11	54	F	NGC 5705.
151	14 35 36	+ 0 06	54 x11	?	B	NGC 5713. Spiral but v. peculiar.
152	14h36m20s	+ 0°01′	33″x11″	104°	B	NGC 5719.
153	14 38 10	+ 0 01	11 x11	30	MB	NGC 5733.
154	15 00 44	+29 59	5 x 5		F	NGC 5840.
155	16 14 34	+15 17	16 x11	?	MF	IC 1209.
156	16 15 00	+11 21	27 x14	155	MB	NGC 6113
157	16 47 30	+45 34	16 x16		MF	NGC 6241.
158	16 57 06	+29 51	11 x11		MB	NGC 6274.
159	16 57 18	+29 57	27 x16		MF	NGC 6282.
160	17 26 09	+60 05	54 x27	25	MF	NGC 6381. Spiral.
161	17 27 22	+60 10	1′ 8 x16	8	F	NGC 6390 (?)
162	17 30 36	+59 41	19 x 8		F	NGC 6399.
163	18 44 54	+45 40	11 x 8		MB	NGC 6702.
164	18 45 14	+45 31	19 x19		B	NGC 6703.
165	20 59 46	+29 31	1 08 x16	173	MF	NGC 7013.
166	22 59 41	+29 40	x 5	. .	MF	NGC 7473.
167	23 11 41	+15 21	54 x14	76	MB	NGC 7567.

Notes.

N.G.C. 476. Not found.

N.G.C. 588, 592, 595, 604, involved in N.G.C. 598 = M. 33.

N.G.C. 1141–2. Not found.

N.G.C. 4572. Can this be identical with I.C. 802?

N.G.C. 5204. This may be identical with No. 627 in Table I. If so the N.G.C. position is incorrect in α.

N.G.C. 6274. This has been identified with No. 158 in Table II. Nothing found in N.G.C. position 1^m west.

N.G.C. 6390. Probably identical with No. 161 in Table II. N.G.C. position probably in error.

N.G.C. 6393, 6394. Cannot be found unless identical with Nos. 789 and 791, respectively, in Table I.

N.G.C. 6610. Not certainly present.

N.G.C. 6702, 6703. N.G.C. positions apparently incorrect. From the descriptions they have been identified with Nos. 163 and and 164 in Table II.

N.G.C. 7551. Not found. Double star in this position.

N.G.C. 7738–9. Not found.

I.C. 131, 132, 133, 134, 135, 137, 139, 140 and 143 are all involved in N.G.C. 598 = M. 33.

I.C. 419. This nebula was discovered by Wolf (A.N., 3130), on a plate taken with a 6-inch lens. He describes it as "Ein heller, wenige Minuten langer, schmaler Nebelfleck." Not found on Mt. Wilson plates. There is a row of small stars extending in an east-west direction. Length of row somewhat over 1′.

I.C. 2733, 2789, 3091. Not found. Double star in each position.

I.C. 2755, 2805. Not found. Faint star in each position.

W VIII, 372. Not found $11^h56^m11^s$, $+29°30'$.

W IX, 67, 95, 110, 113, 125. Not found.

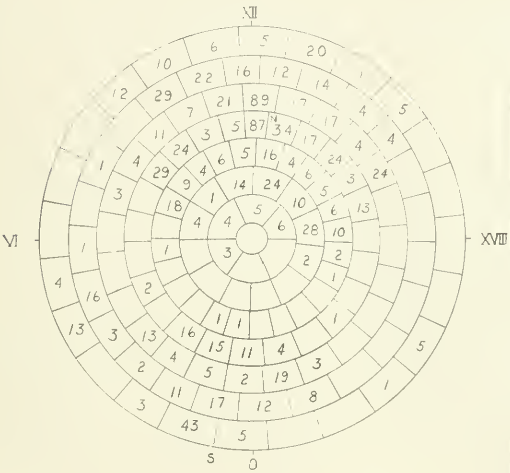
W IX, 79, 118. Not certainly present.

W IX, 147. Appears connected with N.G.C. 5000.

Discussion.

Distribution. The distribution of the nebulæ has been the subject of many investigations and there is little or no new material added here. The figure shows the results obtained. It represents the sky from the north pole to 22°.5 south declination. The circles are 15° apart so that any point half way between is at declination 75°,

60° . . . − 15°. Right ascension is indicated along the circumference. The center of each area thus represents the corresponding "Selected Area." The numbers give the number of nebulæ found on the respective plates. The north and south galactic poles are indicated by N and S respectively. It is evident from the figure that the number of nebulæ in a region does not depend wholly on the galactic latitude. The clustering around the northern pole is much more marked than around the southern.



Orientation. Practically all the nebulæ found on these plates are elliptical in outline, the ellipticity varying from a circle to a narrow line. Most of them are too small to show any details of structure. It does not seem unreasonable to suppose that a large proportion of them are spiral for their appearance could be duplicated in every respect by mere reduction in size of known spiral nebulæ. The spiral nature can be detected with certainty in the larger objects only but the well-authenticated cases pass so gradually to smaller and smaller ones in which less and less structure is evident that the probability that these small elliptical forms are merely unresolved spirals appears very great. An assumption is implied in the above, namely, that most of the material of a spiral nebula lies in one plane and that when the line of sight is perpendicular to this plane the general outline is approximately circular. On this assumption an effort was made to determine whether the planes of the nebulæ showed any preferential orientation with respect to any plane. None could be found.

Number of nebulæ. On the one hundred and thirty-nine plates of this series there were found one thousand thirty-one nebulæ, an average of 7.4 nebulæ per plate. It would require nearly 22,000 plates of the size used to photograph the entire sky. On the assumption that the areas photographed are fair samples this means that the sixty-inch reflector, with exposures of one hour on Lumière Sigma plates, would be able to record 162,000 nebulæ. This number could undoubtedly be increased with longer exposures but to what extent it is difficult to say.

The great difference between the above estimate and the half million of Dr. PERRINE* from the one hundred and four plates of the KEELER program with the Crossley reflector of the Lick Observatory made it appear advisable to attempt to learn the cause of the difference. The average exposure on the Lick plates was approximately three hours. Seed "27" plates were used. Through the kindness of Director CAMPBELL and Dr. H. D. CURTIS of that observatory a similar plate was exposed for three hours on "Selected Area" 56. Dr. CURTIS marked 48 nebulæ on the Crossley plate while I had marked 49 in the same angular area on the 60-inch plate. The large mirrors of both telescopes were not in the best condition when the corresponding plates were taken. Another comparsion was possible for the region surrounding the spiral nebula *N.G.C.* 4254. The Crossley plate for this region was exposed $3^h 19^m$. Within the areas common to the two plates there are thirty-three faint nebulæ on the Lick list† and thirty-four on the Mt. Wilson list. On the strength of these two comparisons it appears that for faint nebulæ an exposure of about three hours with the Crossley reflector on a Seed "27" plate is approximately equivalent to an exposure of one hour with the 60-inch reflector on a Lumière Sigma plate. This does not seem improbable since 1. The Lumière plate is approximately twice as rapid as the Seed plate; 2. Owing to the difference in angular aperture of the two instruments the intrinsic brightness of the image of a surface at the focus of the 60-inch is greater than that of the Crossley reflector in the ratio $\left(\frac{5.8}{5.0}\right)^2$; 3. The advantage of brilliancy of image because of the failure of the photographic reciprocity law. If this conclusion is correct the two series of plates are directly comparable, or, we might say that if the KEELER areas had been photographed with the 60-inch instrument Dr. PERRINE's estimate of 500 000 nebulæ would be duplicated. It was therefore necessary to seek some other cause for the difference in the estimated numbers.

The point next in order was to consider the correctness

* Lick Observatory *Bulletins*, 3, 47, 1904.

† Lick Observatory Publications, Vol. VIII.

of the assumption upon which both estimates are based. This assumption is that the plates of each series are so distributed that they can be considered fairly representative of the entire sky. In order to test this the positions of the centers of the plates were plotted on a small chart of the sky. A difference between the two series quickly came to light. Approximately thirty-three per cent. of the Crossley plates are located within 45° of the north galactic pole while less than twenty per cent. of the "Selected Areas" are found within the same region. Since there can be little question that the plates of the latter series are quite uniformly distributed over the sky it is evident that the former series is not. The relatively large proportion of the Crossley plates in the neighborhood of the north galactic pole, where there is so marked a condensation of nebulæ, is undoubtedly a great if not the determining factor in producing the difference in the two estimates of the probable number of nebulæ in the sky.

There is another factor which should be considered but it is difficult to form any estimate of its importance without additional observations. Dr. KEELER's apparent purpose when making out his observing program was to photograph the larger nebulæ and clusters. The question arises whether there are more or less smaller nebulæ near the larger ones than in fields somewhat removed from the latter. In Dr. PERRINE's article on the number of nebulæ (*loc. cit.*) he states that more nebulæ were found on plates showing no large objects than on the others. It seems possible that this may be due to the fact that the plates showed only about 0.8 square degrees. The area outside the larger objects was therefore rather small and should not be compared directly with the total area when only small objects were photographed. The only "Selected Area" plates which throw any light on this matter are the two containing *N.G.C.* 598 and 4254. The first contains fifteen nebulæ while the four nearest it on the north, south, east and west average 8. The second shows eighty-eight while the average of the surrounding four is thirty-six. These averages would be of greater value if they were obtained from plates within 5° of the two central ones, but since none of this nature are available the evidence must be taken for what it may be worth. It at least indicates another possible reason for the difference mentioned.

SUMMARY.

The results of this investigation may be summed up as follows:

1. There is a great condensation of nebulæ toward the north galactic pole and a similar but much less marked one toward the south galactic pole.
2. On the assumption that most of the nebulæ are approximately disk-shaped the planes of the disks appear to be oriented at random in space.
3. The probable number of nebulæ in the sky which could be photographed with the 60-inch reflector with exposures of one hour on Lumière Sigma plates is 162,000.
4. 864 new nebulæ were discovered.

Smith Observatory, Beloit, Wis., July 25, 1913.

ADDENDUM.

Professor WOLF's *Koenigstuhl-Nebel-Listen* 13 and 14 have been recently received at this observatory. Seven objects from Table I are represented in the above lists. The following identifications appear probable: 1 is *W* XIII, 108, and 83, 89, 91, 94, 95 and 96 are *W* XIV, 169, 197, 202, 217, 225 and 234 respectively.

1913, *November* 24.

ON A TEMPERATURE GRADIENT TERM IN THE COLLIMATION CONSTANT OF THE ALBANY MERIDIAN CIRCLE,

By SEBASTIAN ALBRECHT.

Within recent years quite elaborate precautionary measures have been taken to insure accuracy in some of the constants employed in the reductions of meridian circle observations, with special reference to fundamental work. This is especially true of the clock-rate, by means of the temperature and pressure controlled Riefler clocks, and of the azimuth of the transit-instrument, by such work as that of GILL and HOUGH at the Cape Observatory. The level is usually determined by several observations on the nadir as part of the observing program for each day. In this way, however, a knowledge of the level of the instrument is tied up with the collimation. The collimation is quite generally considered subject to little change, and for this reason it is determined at much longer intervals, usually from one to several weeks.

For the meridian circle of the Dudley Observatory the collimation varies linearly as a function of the temperature.* In view of this direct dependence of the collima-

*CARNEGIE Institution of Washington, Year Book 11, p. 166, 1912.

tion on the temperature, it occurred to the writer that there might also be an effect depending upon the rate at which the temperature is changing at the time of observation.

A test for such an effect was made with the collimations observed at Albany since the return of the instrument from San Luis. The observations, mostly by Mr. Roy and some by Mr. Varnum, were made by observing, Clamp E. and Clamp W., on the mire and on the mercury collimator underneath the telescope. In the discussion which follows the collimations thus obtained will be referred to as the "mire" and "nadir" values. Corrections for pivot-errors were not applied as these do not affect the results obtained below.

The observed collimations were first plotted — separately for mire and nadir — with collimations as abscissæ and temperatures T as ordinates. The plots show the temperature effect clearly and unmistakably. A straight line represents well the smoothest curve that can be drawn through a group of observations extending over an interval of time during which no discontinuity occurred in the collimation due to intentional or accidental changes in the instrument. The slope of the lines thus drawn, which was found to be identically the same for nadir and mire, gives the temperature coefficient of the collimation, *i. e.*, the change of collimation per degree Centigrade increase in temperature. Between the limiting temperatures of the observations, $-16°$ and $+34°$C, this was found to be $-0^s.005$ (Clamp E.), which is in exact agreement with the value determined previously by Professor Boss (*l. c.*) for the same instrument mounted at San Luis.

The graphs described above furnish a ready means for making a preliminary test for a temperature gradient effect, as follows: The straight lines drawn to best represent the points in each group furnish a means for obtaining the computed collimation for each temperature on the basis of a linear relation between the collimation and the temperature. The horizontal distances of the individual points from the correponding straight line give residuals which should represent the accidental errors of observation if a linear temperature effect alone is involved. These residuals were separated into two divisions, corresponding to forenoon and afternoon observations respectively. For the forenoon observations, on the average, the temperature would be rising. As the afternoon observations were usually taken at about half-past three o'clock or later, the temperatures for these observations, on the average, would have a downward trend. Table I gives the residuals thus separated. The concentration of the negative residuals in the A. M., and of the positive residuals in the P. M. columns, is quite marked, and indicates a dependence of the collimation upon the direction in which the temperature is changing at the time of observation.

Table I.

Ref. No.	Δc, observed-computed			
	A. M.		P. M.	
	nadir	mire	nadir	mire
Group 1*	0^s.001			
1	−5	−25	...	
2	...	−28	...	
3	−20	−30	...	
4	...	+15	...	
5	+28	+29		
6	...	...	...	+35
Group 2				
7		−29		
8			−2	+7
9				+1
10			−18	−18
11	−4	−10	...	...
12	...	...	+10	−9
13		...	+14	+9
14		...	+4	+5
15			+19	+21
16			−23	−22
17	−5	−27	...	...
18			...	+28
19	−22	−29		...
20	...	...	+24	+16
Group 3				
21				+51
22	+16	+9		
23	+23	...		
24	−21	−13		
25	−25	−34		
26	...	...	+74	+43
27	−16	−5	...	...
28	...	...	+35	+54
29	...	...	+62	+37
30	−40	−37	...	...
31	−30	−34	...	...
32	...	...	+3	+25
33	−64	−72	...	
34	...	...	+7	+5

The material was investigated in greater detail as follows It had previously been found (*l. c.*) that there is a lag in the instrumental constants. Accordingly, the temperatures recorded with the observations for collimation were those of the "attached thermometer," the barometer being enclosed in a wooden case in the observing room so as to introduce a lag, and thus give a temperature which

*Group 1, from November 8 to December 28, 1911. Group 2, from January, 1912 to February 13, 1913. Group 3, from April, 1913 to October 6, 1913. Between groups the collimation was intentionally or accidentally changed. The signs in this paper will refer to Clamp E.

might more nearly correspond to the temperature of the telescope. As it was feared that these temperatures (T) might to a certain extent mask the effect sought for, and in order to have a more complete record of the air temperatures within several hours of the times of observation, use was made of the temperatures from the records of the Albany station of the U. S. Weather Bureau. These were kindly placed at my disposal by Professor TODD. These, to be refrrred to as γ, are not strictly the temperatures of the air surrounding the telescope, but they are the best approximations obtainable, and as we shall see later, lead to reliable results.

The collimation may be expressed in terms of the equation

$$c = A_{\text{Group}} + Bt + \Delta c \qquad (1)$$

where A_{Group} and B are constants which can be computed or taken directly from the graphs described above, t is the temperature, and Δc the temperature gradient term. In this case it was found that t could be replaced by either T or γ with similar final results. However, in general it will be more satisfactory to endeavor to obtain the temperature of the air surrounding the telescope.

It is evident that equation (1) may also be put into the form

$$c = A_{\text{Group}} + B\,(t + \Delta t) \qquad (2)$$

since it will always be possible to find a Δt such that $\Delta c = B \cdot \Delta t$. In practical use the second form of the equation is perhaps slightly the more convenient, and the solution for the temperature gradient term was made in this form.

To facilitate the solution, the auxiliary angle ψ is introduced, where $\Delta t = f\,(\psi)$. This function is to be determined. The hourly temperatures γ within several hours of the time of an observed collimation are plotted on any convenient scale with temperatures increasing upward as ordinates and times increasing toward the right as abscissæ. A smooth curve is drawn through the points. At the point on the curve corresponding to the observed time of collimation a tangent line is drawn. The angle ψ which this line makes with the x-axis is measured with a protractor. It is called positive when the tangent line is sloping upward and negative when sloping downward toward the right. In this way an angle ψ is determined for each observed time of collimation.

Residuals ΔT, corresponding to the residuals Δc in Table I, were taken directly from the original graphs. These were arranged progressively in the order of magnitude of the angles ψ, with due regard to the signs of the angles, and the normal places given in the first four columns of Table II were formed. The $f\,(\psi)$ is to be determined of a form which shall be as simple as possible for computation and still retain all the accuracy warranted by the character of the data. A graph was made from the data in Table II with ΔT as ordinates and ψ (degrees used simply as numbers) as abscissæ. A smooth curve was drawn to best represent the points plotted. By actual trial it was found that an equation of the form

$$\Delta T = a + b\,.\,\tan \tfrac{1}{2}\psi \qquad (3)$$

gives a curve which is practically coincident with the curve described above over a length corresponding to the upper five-sixths of Table II. The last value alone in the table is at a considerable distance from the computed curve, and this point has a small weight.

TABLE II.—THE TEMPERATURE GRADIENT TERM.

ψ	$T\Delta$	No. of Obs.	Total Weight	Natl. $\tan\frac{1}{2}\psi$	$\Delta T = -1.67 + 19.3\tan\frac{1}{2}\psi$	$\Delta c = -0^s.0043\Delta T$
°	°				°	s
+55.1	+8.6	6	16	+0.52	+8.4	−0.036
+39.8	+3.9	7	19	+ .36	+5.3	−0.023
+29.3	+5.1	7	17	+ .26	+3.4	−0.015
+18.7	+1.7	11	16	+ .16	+1.4	−0.006
± 0.0	−3.3	10	23	± .00	−1.7	+0.007
−10.4	−2.6	8	12	− .09	−3.4	+0.015
−18.1	−5.2	6	10	− .16	−4.8	+0.021
−33.3	−3.5	3	4	− .30	−7.5	+0.032

Each line in Table II gives an equation of the form (3), with the constants a and b as the only unknowns. Solving by least squares we obtain the values of these constants, and substituting in (3) we get

$$\Delta T = -1.67 + 19.3 \tan \tfrac{1}{2}\psi \qquad (4)$$

Column six in Table II gives the values of ΔT computed from (4), and the last column gives these values converted into the corresponding corrections to the collimations.

The method by which the temperature gradient term was determined is sufficiently illustrated above. The solution was repeated with the use of γ and $\Delta\,\gamma$ instead of T and ΔT. The following formulæ are provisionally adopted.

$$c = A_{\text{Group}} - 0^s.0043\,(\gamma + \Delta\gamma) \qquad (5)$$

where

$$\Delta\gamma = -1.95 + 17.6 \tan \tfrac{1}{2}\psi \qquad (6)$$

The constants in (6) apply only for the particular scale used in the graphs. On the quadrille-ruled paper employed, one hour was equal to two divisions horizontally, and two degrees Fahrenheit were equal to one division vertically.

Table III is self explanatory. The probable errors in groups 1, 2, and 3*a* do not involve the rejection of a single observed collimation. On line 3*b* are given the probable errors for group 3 obtained by excluding four abnormally large residuals. Combining the results from nadir and mire, fifty-four individual values, into one value, we obtain a probable error of $\pm 0^s.0145$ when the collimations are corrected for temperature only, and $\pm 0^s.0096$ when corrected for both temperature and temperature gradient. With the temperature gradient term included the probable error of an observed collimation was reduced to sixty-seven per cent. of its value when this term was omitted.

Table III. Probable Errors of an Observation for Collimation.

Group	When Corrected for Temperature only				When Cor. for both Temp're and Temperature Gradient			
	nadir	No. of Obs.	mire	No. of Obs.	nadir	No. of Obs.	mire	No. of Obs.
	s		s		s		s	
1	±0.015	3	±0.017	6	±0.004	3	±0.006	6
2	.009	10	.011	13	.011	10	.010	13
3*a*	.023	13	.023	13	.018	13	.015	13
3*b*	.018	10	.019	12	.010	10	.011	12
1, 2, 3*b*	±0^s.0145			54	±0^s.0096			54

To apply the temperature gradient term in practice, the temperatures are simply plotted on quadrille-ruled paper on the same scale as in the original solution, the tangent line drawn at the point on the curve corresponding to the time for which the computed collimation is desired, and the angle ψ measured with a protractor; the value of ψ is substituted in (6), and this equation is solved for $\Delta\gamma$. This is done very quickly with the use of the natural tan $\frac{1}{2}\psi$ and a slide rule or Crelle's Tables. The whole operation can be performed well within five minutes.

It would be of interest to have a test made for these effects in other transit instruments. The $f(\psi)$ need not necessarily be a trigonometric function. In the more general case, with ψ used as simply representing numbers, the $f(\psi)$ may be taken in the form of the series

$$f(\psi) = \alpha + \beta\,\psi + \gamma\,\psi^2 + \ldots \tag{7}$$

In fact, judging from the shape of the curve plotted from the data in Table II, it seems likely that a parabola

$$f(\psi) = \alpha + \beta\,\psi + \gamma\,\psi^2 \tag{8}$$

would have given an equally good, if not better, representation of the points. The uncertainties in the data at present available make it inadvisable to carry the refinement of computation further. When conditions become favorable, an effort will be made to obtain a nearly continuous series of collimations in order to determine, as far as possible, the real nature of the $f(\psi)$ for the Albany meridian circle. It is not impossible that this function may be continuous only within more or less definite sets of limits, *i. e.*, there may be present discontinuities analogous to a certain extent to lost motion in a screw.

It may not be out of place to discuss briefly the possible bearing of the above results on meridian work.

The period of the temperature gradient term should be quite definitely a day, corresponding to the more or less regular diurnal change of temperature. Since the average daily temperature curves differ somewhat for the different seasons, as regards shape, amplitude, and position of maximum and minimum, there is probably an annual period superimposed upon the diurnal period.

Star positions, determined with instruments which have temperature gradient effects in the collimation, will fortunately not contain the complete effect of this term. The periods involved are subject to considerable minor irregularities. A distribution of the observations over most of the night and to a minor extent over several seasons reduces the mean effect as a systematic term; moreover it will be eliminated to a large extent by observing all stars an equal number of times Clamp E. and Clamp W. In a favorable case the final right ascensions may be almost entirely free from this source of error. A very unfavorable case would be one in which, possibly on account of greater convenience, all of the star observations have been made in the early part of the night, *i. e.*, with a descending temperature, and where all of the observations for collimation have been obtained during the part of the day when the temperature is rising, which is in the forenoon or before two o'clock in the afternoon. In the latter case the collimations used in the reductions of the star transits would be systematically erroneous by practically double the temperature gradient term, because this term would be of opposite signs for the observations for collimation and for the star observations respectively. If such an observing practice were combined with very unequal numbers of observations in the two positions of the instrument, the final right ascensions would be quite seriously in error. Fortunately modern star catalogues are based upon an equal number of observations in the two positions of the instrument, and therefore the right ascensions are liable to only small residual systematic errors due to incompletely determined collimations.

We can readily see how vital may be the effect in fundamental work, where observations extend through the twenty-four hours of the day, and consequently are subject to the complete diurnal change in the temperature gradient term. Circumpolar stars observed for azimuth in the early part of the evening and again twelve hours

later in the forenoon, would contain this term with opposite signs. The possible magnitude of the systematic error thus introduced into the work for the day may be inferred from the values of the temperature gradient term given in the last column of Table II. Fortunately in this case also an equal number of observations in the two positions of the instrument will eliminate the major part of the errors from the definitive star positions. However, in fundamental work the remaining residual errors of both a systematic and accidental nature are a more serious concern.

Considerable attention has been given to the determination of outstanding residuals of a systematic nature in the various star catalogues, and most of these remain unexplained.

In general, depending somewhat upon the details of the methods employed in the observations and reductions, the adopted meridian for a night's work is forced into coincidence with the true meridian near the equator and near the pole, while between and below these points the two curves will be separated. If on another night the same star list be observed with the telescope reversed but with conditions the same in all other respects, then the means for the two nights will be free from systematic error from this source. Such a complete balancing of errors will rarely occur in practice.

Where a temperature gradient effect is contained in observations to any considerable extent, it should become evident when the observations are separated into the divisions Clamp E. and Clamp W., for stars approximately midway between the pole and the mean of the declinations of the time stars used. This separation is generally not possible for star catalogues without recourse to the original data. The published tables of the systematic corrections $\Delta\alpha_\delta$, do not enable us to infer much in regard to the source of this error, except that it may be of interest to note the general tendency to zero values near the pole and near the equator, and to large values between these points. Also the practical absence of this error in the LISBON catalogue (1890) may have some bearing, as the collimation was entirely eliminated from the observations for this catalogue by reversal of the instrument on each star.

For the *Cordoba General Catalogue* (1875) the separation into Clamp E. and Clamp W. is possible, and has been made by Professor Boss. In Boss' *Preliminary General Catalogue*, p. 339, is given the table of corrections $\Delta\alpha_\delta$ thus separated. It is reproduced in Table IV of this paper, and the column "corr. × cos δ" has been added in order to show more clearly the progressive increase toward the zenith from both the pole and the equator. The effect here shown could quite clearly be produced by more or less systematically erroneous values of the collimations.

TABLE IV, (Cordoba, 1875).

δ	corr. (Cl. E.)	corr. × cos. δ
± 0°	−0.006s	−0.006s
− 5	− .001	− .001
10	+ .004	+ .004
15	+ .009	+ .009
20	+ .014	+ .013
25	+ .018	+ .016
−30	+ .023	+ .020
35	+ .027	+ .022
40	+ .028	+ .021
45	+ .029	+ .020
50	+ .030	+ .019
−55	+ .030	+ .017
60	+ .030	+ .015
65	+ .030	+ .013
70	+ .030	+ .010
75	+ .030	+ .008
−80	+ .030	+ .005
−85	+ .030	+ .003

TABLE V. COLLIMATIONS AT THE CAPE OBSERVATORY.

Date	Mean for Month	No. of Obs.	Extremes lowest	Extremes highest	Total Range
1880, April	+0.031s	19	+0.010s	+0.051s	0.041s
May	+ .029	19	− .005	+ .049	.054
June	+ .047	21	+ .028	+ .072	.044
July	+ .049	17	+ .024	+ .073	.049
Aug.	+ .045	16	+ .028	+ .062	.034
Sept.	+ .043	17	+ .028	+ .058	.030
Oct.	+ .036	22	+ .011	+ .060	.049
Nov.	+ .027	26	− .037	+ .060	.097
Dec.	+ .029	15	+ .006	+ .058	.052
1881, Jan.	− .001	15	− .024	+ .025	.049
Feb.	+ .008	23	− .016	+ .029	.045
Mar.	+ .011	27	− .012	+ .037	.049
Apr.	+ .012	17	− .009	+ .034	.043
May	+ .033	12	+ .007	+ .070	.061
June	+ .037	23	+ .010	+ .069	.059
July	+ .039	22	+ .020	+ .059	.039
Aug.	+ .028	20	− .009	+ .048	.057
Sept.	+ .027	23	+ .010	+ .049	.039
Oct.	+ .021	21	− .004	+ .036	.040
Nov.	+ .015	21	− .017	+ .030	.047
Dec.	+ .001	18	− .035	+ .020	.055
1882, Jan.	± .000	21	− .024	+ .030	.054
Feb.	− .005	29	− .062	+ .017	.079
Mar.	+ .009	24	− .014	+ .032	.046

Table V.—*Continued.*

Date	Mean for Month	No. of Obs.	Extremes lowest	Extremes highest	Total Range
	s		s	s	s
1882, Apr.	+ .008	17	− .025	+ .035	.060
May	+ .017	17	− .033	+ .045	.078
June	+ .010	10	− .010	+ .015	.025
July	+ .016	16	− .008	+ .043	.051
Aug.	+ .012	9	− .015	+ .023	.038

Table VI. Nov. 25, 1880.

Collimation	Observer
+0.001	R. T. Pett
+0.012	G. W. H. Maclear
+0.035	David Gill (?)
+0.037	David Gill
+0.060	W. H. Finlay
+0.051	R. T. Pett

An examination was made of the separate collimations published with the Cape Meridian Observations, to see if they might have a bearing upon the problem discussed in this article. In Table V is given a summary of that part of the material* which contains a large number of observations in each month. The monthly means in column two show quite clearly a variation with a period of a year. The collimations for the two years following the table have been examined and show the same periodicity, with the maximum near the end of June and a minimum in January, thus giving five consecutive years. The inference of a temperature effect seems quite clear. With the presence of a temperature effect, together with the large total range between the individual values (summarized in columns four, five, and six of Table V), a temperature gradient term may also be suspected. Also, in the absence of any other indicated cause, the large range between the six observations on one day, reproduced in Table VI, is suggestive of the latter effect, though other causes are not necessarily excluded.

*Cape Meridian Observations 1879-81, pp. 5-9, and 1882-84, pp. 3-4.

It seems likely that, for the numerous transit instruments in use, a variation of the collimation as a function of the temperature* will be found to be the rule rather than the exception; and where a definite variation with the temperature exists there is also apt to be an effect depending upon the rate at which the temperature is changing. This fact should by no means reduce our confidence in the transit instrument. A variation according to definite laws which can be established is much more satisfactory than variations of an irregular and apparently arbitrary nature. The material reduction in the probable error of the so-called "collimation constant," and the increased certainty as to the proper values to be used in the reductions of star observations, should certainly increase our confidence in the instrument rather than diminish it.

We are not to infer from a varying collimation that we must observe for the collimation more frequently than in the past. Quite the contrary may be found to be true. For each instrument the laws governing the variations must first be established by the requisite number of observations, and thereafter only a sufficient number of observations need be obtained to give a good control over the formulæ. With due regard to a proper distribution of the observations, as regards high and low temperatures, upward and downward trend, and the stationary points, an average of one determination per week should prove sufficient.

*A tabulation of the collimations for the Greenwich Observations for 1909 and 1910, shows likewise a definite variation with the period of a year, and with a double amplitude of about $0^s.13$.

Dudley Observatory, Albany, N. Y., December 18, 1913.

OBITUARY NOTICE.

Dr. Seth C. Chandler, formerly editor of the *Astronomical Journal*, and more recently associate editor, succumbed to an attack of pleurisy on December 31, 1913, at his home in Wellesley Hills, Mass. A sketch of Dr. Chandler's career will follow in the next number.

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HYPERGEOMETRIC SERIES AND WALKER'S TABLES OF THE LEVERRIER COEFFICIENTS.

By G. W. HILL.

Nearly all the series employed in astronomy have the form

$$\mu_0\, \alpha^m\, [1 + \mu_1\, \alpha^2 + \mu_1\, \mu_2\, \alpha^4 + \mu_1\, \mu_2\, \mu_3\, \alpha^6 + \ .\ .\ .\ .\ ]$$

where α^2 is the parameter according to which the series is arranged, μ_i is the factor by which the $(i-1)^{\text{th}}$ coefficient must be multiplied to produce the i^{th} coefficient and m is an exponent generally integral.

We propose to transform this series by multiplying it by the unit factor $\dfrac{1-\alpha^2}{1-\alpha^2}\cdot$ The result is

$$\frac{\mu_0\ \alpha^m}{1-\alpha^2}\Big[1+(\mu_1-1)\alpha^2+\mu_1(\mu_2-1)\alpha^4+\mu_1\mu_2(\mu_3-1)\alpha^6+\ \ .\ .\ .\ \Big].$$

Therefore, if we put

$$M_1 = \mu_1 - 1, \quad M_2 = \mu_1\frac{\mu_2-1}{\mu_1-1}, \quad M_3 = \mu_2\frac{\mu_3-1}{\mu_2-1}, \quad \text{etc.,}$$

we have

$$\frac{\mu_0\ \alpha^m}{1-\alpha^2}\Big[1 + M_1\, \alpha^2 + M_1\, M_2\, \alpha^4 + M_1\, M_2\, M_3\, \alpha^6 + \text{etc.}\ .\ .\ .\ \Big]$$

of like form as before. This operation can be repeated. Thus, putting

$$N_1 = M_1 - 1, \quad N_2 = M_1\frac{M_2-1}{M_1-1}, \quad N_3 = M_2\frac{M_3-1}{M_2-1}\ .\ .\ .$$

we have

$$\frac{\mu_0\ \alpha^m}{(1-\alpha^2)^2}\Big[1 + N_1\alpha^2 + N_1\, N_2\alpha^4 + N_1\, N_2\, N_3\, \alpha^6 + \ \ .\ .\ .\ \Big].$$

Should any of the μ be exactly equal to unity, zero divisors would appear in the values of M. But if any one of the μ is zero, the series passes from an infinite to a finite series ending with the appearance of μ_{i-1}. But, if one of the μ is exactly unity as, for instance, μ_3, we have

$$\frac{\mu_0\ \alpha^m}{1-\alpha^2}\Big[1 + (\mu_1-1)\,\alpha^2 + \mu_1\,(\mu_2-1)\,\alpha^4 + 0\alpha^6 + \mu_1\,\mu_2\,\mu_3\,(\mu_4-1)\,\alpha^8 + \mu_1\,\mu_2\,\mu_3\,\mu_4\,(\mu_5-1)\,\alpha^{10} + \ \ .\ .\ \Big]$$

The difficulty with this series is tided over if we conceive that in $M_3 = \mu_2\dfrac{0}{\mu_2-1} = 0$, and $M_4 = \mu_3\dfrac{\mu_4-1}{0}$, the zeros are identical and thus $M_3\, M_4 = \mu_2\, \mu_3\ \dfrac{\mu_4-1}{\mu_2-1}$; thus the intrusion of ∞ into the series is prevented.

Of the immense variety of series comprehended in our general form, each one is sufficiently characterized by a statement of the mode of construction of the μ. We limit ourselves to the case where μ_i is a function of i. The species of hypergeometric series especially interesting to astronomers is that where μ_i as a function of i is exemplified by the equations

$$\mu_1 = \frac{\alpha\beta}{\gamma}, \quad \mu_2 = \frac{(1+\alpha)\,(1+\beta)}{2\,(1+\gamma)}, \quad \mu_3 = \frac{(2+\alpha)\,(2+\beta)}{3\,(2+\gamma)}, \quad \mu_4 = \frac{(3+\alpha)\,(3+\beta)}{4\,(3+\gamma)}, \quad \mu_{i+1} = \frac{(i+\alpha)\,(i+\beta)}{(i+1)\,(i+\gamma)}.$$

α, β, γ are constant for the same series but may vary from one series to another. We shall limit our investigation to the function $b_{\frac{1}{2}}^{(j)}$ of LAPLACE and its derivatives with respect to the parameter α. The subscript ½ will be omitted as unnecessary. Here the α and β are the halves of odd integers and γ a non-negative integer.

As it is inconvenient to have the half integers in the expression of μ_i we multiply both numerator and denominator by 4. For $b^{(j)}$ the constants have the values $\alpha = \frac{1}{2}$, $\beta = j + \frac{1}{2}$, $\gamma = j + 1$. Thus here μ_i has the value

$$\mu_i = \frac{(2i-1)(2i+2j-1)}{2i(2i+2j)} = \left[1 - \frac{1}{2i}\right]\left[1 + \frac{2j-1}{2i}\right]\left[1 + \frac{j}{i}\right]^{-1}$$

From this equation it is perceived that μ_i cannot vanish while i is going from unity to ∞ and also that $\mu_\infty = 1$. After i has received a value somewhat surpassing j, μ_i continually approximates unity.

If we use n to denote order of the derivatives of $b^{(j)}$, it will be seen, by the cancellation of factors common to the numerator and denominator, that the μ_i for $\alpha^n D^n b^{(j)}$ is

$$\mu_i = \frac{(2i-1)(2i+j-1)(2i+j)(2i+2j-1)}{2i(2i+j-n-1)(2i+j-n)(2i+2j)}.$$

This, however, is a general formula, it is alway true, but i must be counted from the first term of $b^{(j)}$, whether it falls out through the derivation, or not. But it is convenient to make $i = 0$ for the first significant term in the derivative. In WALKER's table the number of terms which fall out is indicated by the number of vacant places at the beginning of the line.

The method of LEVERRIER and after him of WALKER was to improve the series for the calculation of $\alpha^n D^n b^{(j)}$ by multiplying it by $(1-\alpha^2)^n$. This factor seems to have been chosen simply as the result of trial. But one may exhibit the benefit of such a multiplication by bringing forward a very simple case. Let it be proposed to evaluate $(1-\alpha^2)^{-\frac{1}{2}}$ by employment of the binomial theorem. By direct use of this the coefficient of α^{20} is $\frac{15.17\;\;33}{2.4\,.\,.20}$, but if we consent to write the expression thus $(1-\alpha^2)^{-1}(1-\alpha^2)^{\frac{1}{2}}$ and develop only the latter factor we have a coefficient of the same power of α $\frac{3.5}{2.4}\;\frac{17}{20}$. The ratio of the two coefficients is $\frac{17678835}{13}$; thus, the advantage of the latter course is relatively enormous. Nevertheless, the pure mathematician regards both series as equally convergent, the limit of convergence in both being the same, viz., $\alpha = 1$.

WALKER employs the symbol β which is the tangent of the angle of which α is the sine. Here we must explain a peculiarity in the arrangement of his tables. By removing as a factor the proper power of β^2 it will be perceived that, when $n = 2$, he has a term in β^{-2}; when $n = 3$, an additional term in β^{-4}, and so on till when $n = 7$, he has terms in β^{-14}, β^{-12}. . β^{-2}. We have

$$\alpha^{n+2-j} D^n b^{(j)} = \alpha^{-2m}\beta^{2n}[1 + h\alpha^2 + \ldots]$$

where m is 0 or a positive integer. Now WALKER, whenever no negative powers of α^2 would thereby be introduced, always makes the multiplication involved in the second members of these equations by employing the first factors just as they stand; but when negative powers of α^2 arise, he eliminates them by menas of the relation $\alpha^{-2} = 1 + \beta^{-2}$, which gives

$$\alpha^{-2}\beta^4 = \beta^2 + \beta^4, \qquad \alpha^{-4}\beta^6 = \beta^2 + 2\beta^4 + \beta^6, \qquad \alpha^{-6}\beta^8 = \beta^2 + 3\beta^4 + 3\beta^6 + \beta^8, \text{ etc. . . .}$$

However, with WALKER there is no extra computation involved in this elimination, as the proper coefficient is found by proceeding in a diagonal line leading upwards through his columns of differences. In some cases, it is necessary to prolong the columns of difference upward by adding a certain number of zeros to the main column. This it is never necessary to do if one uses the method of multiplying by $1-\alpha^2$ the necessary number of times in succession.

The method of computation followed by WALKER was purely arithmetical. He first computed the coefficients of the developments of the b to eighteen places of decimals (LEVERRIER has given them to twelve, *Annals de l'Observatoire de Paris*, Tom. II). Then differentiated the expressions seven times in succession, then differenced the sets of coefficients only once for the first derivatives but going up to seventh differences for the seventh derivatives. Then his work was about concluded, but he had still to get the logarithms to be inserted in his tables. Thus, although each coefficient is rigorously equal to a vulgar fraction, WALKER had no information on the terms which compose it.

The sheets containing WALKER's computations were, for some time, preserved in the office of the American Ephemeris, but are now lost. An abstract of the results was published in the American Ephemeris for 1857, but, unfortunately, a regard for the breadth of page, led the compiler to reduce all the decimals to five places. This is insufficient for investigations of the perturbations of the solar system. It has seemed to me that the method of WALKER has many advantages over the methods of recursion now generally employed, the chief of which is

that you can derive any coefficients you want without having to deal with any that precede in the natural order. Hence it seemed that a recomputation of these tables would be a service acceptable to astronomers. In this work I have not employed the arithmetical method of WALKER, save occasionally for verification, preferring generalized forms.

The values of the $b^{(j)}$ have been given by LEVERRIER, in their case there is no need of adding anything. For the first derivative, by differentiation, we have

$$\alpha D b^{(j)} = \frac{3.5 \ldots (2j-1)}{4.6 \ldots 2j} \alpha \left\{ j + \frac{1}{2} \frac{2j+1}{2j+2} (j+2) \alpha^2 + \frac{1.3}{2.4} \frac{(2j+1)(2j+3)}{(2j+2)(2j+4)} (j+4) \alpha^4 + \ldots \right\}$$

From which, i being counted from the first significant term,

$$\mu_i = \frac{(2i-1)(2i+j)(2i+2j-1)}{2i(2i+j-2)(2i+2j)} .$$

Let us multiply the series within the brackets by $1-\alpha^2$; the last factors of each coefficient, in their order, are

$$j , \quad \frac{2+j-2j^2}{4(j+1)} , \quad \frac{4+j-2j^2}{8(j+2)} , \quad \frac{6+j-2j^2}{12(j+3)} , \quad \frac{8+j-2j^2}{16(j+4)} , \ldots .$$

The table of factors by which the coefficients of the series for $b^{(j)}$ must be multiplied in order to obtain the coefficients next following in $(1-\alpha^2)\,\alpha D b^{(j)}$ is

j	
0	$\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10} \ldots$
1	$\frac{1}{8}, \frac{3}{24}, \frac{5}{48}, \frac{7}{80}, \frac{9}{120} \ldots$
2	$-\frac{4}{12}, -\frac{2}{32}, \frac{0}{60}, \frac{2}{96}, \frac{4}{140} \ldots$

* * * * * * * *

The rest can be readily filled in from the formula.

For the second derivatives of $b^{(j)}$ I have found the general formula

$$\mu_i^{(j)} = \frac{(2i-5)(2i+2j-5)}{2i(2i+2j)} \frac{f(i)}{f(i-1)} ,$$

where

$$f(i) = (16j^2+20)\,i^2 + (16j^3 - 72j^2 + 20j - 18)\,i + 3j(j-1)(2j-3)(2j-1) .$$

This formula is true without any exceptions, but it must be borne in mind that i is counted from the first term of $b^{(j)}$ whether it disappears by being twice differentiated or not.

After the expansion of $\alpha^2(1-\alpha^2)^2 D^2 b^{(j)}$ has been found by means of the values of $\mu_i^{(j)}$, in order to get the proper value of the second coefficient of WALKER's table we must augment $\mu_i^{(j)}$ by a unit (this is on account of WALKER's virtually beginning the horizontal line with $\beta^{-2} = \frac{1-\alpha^2}{\alpha^2}$ and we recall that $\alpha^{-2}\beta^4 = \beta^2 + \beta^4$.

The third derivatives of the $b^{(j)}$ are naturally more complicated than the second; hence the information about them is less complete. For the first four values of j we have found

$$\mu_i^{(0)} = \frac{(2i-3)^2(14i+3)}{2i(2i+4)(14i-11)} , \qquad \mu_i^{(1)} = \frac{(2i-3)^2(38i+1)}{2i(2i+4)(38i-37)} .$$

$$\mu_i^{(2)} = \frac{(2i-3)(2i-1)(11i-2)}{2i(2i+6)(11i-13)} , \qquad \mu_i^{(3)} = \frac{(2i-3)(2i-1)(23i-1)}{2i(2i+6)(23i-24)} .$$

From $j=4$ to $j=12$, $\mu_i^{(j)}$ has the factor $\frac{(2i-7)(2i+2j-7)}{2i(2i+2j)}$ beside the factor $\frac{f(i)}{f(i-1)}$. The forms of $f(i)$ for the several values of j are

$$\begin{aligned}
j &= 4 & f(i) &= 398i^3 - 1971i^2 + 1573i - 1050 \\
j &= 5 & f(i) &= 1228i^3 - 7968i^2 + 1385i - 15750 \\
j &= 6 & f(i) &= 878i^3 - 7617i^2 - 5735i - 34650 \\
j &= 7 & f(i) &= 476i^3 - 5484i^2 - 10007i - 45045 \\
j &= 8 & f(i) &= 310i^3 - 4659i^2 - 13747i - 60060
\end{aligned}$$

$j = 9 \quad f(i) = 3916i^3 - 7519i^2 - 307399i - 1392300$
$j = 10 \quad f(i) = 2414i^3 - 58029i^2 - 302915i - 1453500$
$j = 11 \quad f(i) = 5836i^3 - 172404i^2 - 1093075i - 5595975$
$j = 12 \quad f(i) = 2082i^3 - 74349i^2 - 553701i - 3018410$

The nine functions denoted by $f(i)$ appear irregular, but this is caused by the throwing out of integral factors in order to reduce $\frac{f(i)}{f(i-1)}$ to its lowest terms. If these are restored, being severally in order 6, 3, 6, 15, 30, 3, 6, 3, 10, being differenced, the coefficients of i^3 show a constant second difference of 288, the coefficients of i^2 a constant fourth difference of -1152, the coefficients of i a constant fifth difference of -5760, and the absolute terms a constant sixth difference of 14400. Thus it is plain that $f(i)$ is an integral function of j.

For the fourth derivatives of $b^{(j)}$ we shall not attempt any general formulæ nor, beyond $j = 2$ any formula for μ_i. For $j = 0$, we have

$$\mu_i = \frac{(2i-5)^2 f(i)}{2i(2i+4) f(i-1)} \qquad f(i) = (26i)^2 - 472i + 27.$$

For $j = 1$, we have

$$\mu_i = \frac{(2i-3)^2 (50i+3)}{2i(2i+6)(50i-17)}$$

For $j = 2$, we have

$$\mu_i = \frac{(2i-5)(2i-3) f(i)}{2i(2i+6) f(i-1)} \qquad f(i) = 874i^2 - 1055i - 6.$$

For the Walker coefficients of $\alpha^n (1-\alpha^2)^n D^n b^{(j)}$ we must refer to the tables given further along. The matter has proved so refractory that the μ_i could not be found as a function of i. Two tables are devoted to this object. The second gives the numerical coefficient of the first significant term of the function $\frac{1}{n!}\alpha^n D^n b^{(j)}$. The first table supplies the means of deriving a certain number of terms in the Walker manner. The process will be readily understood by an example. Suppose we wish to obtain the numerical factors of the terms of $\frac{1}{24}\alpha^4 D^4 b^{(3)}$. In the second table in the column headed $n = 4$, and horizontal line answering to $j = 3$ we get the fraction $\frac{175}{128}$. In the first table under the heading $b_4^{(3)}$, we find a set of numerators in the column headed N, and a set of factors for a denominator headed D. The series represented by its numerical coefficients is

$$\frac{175}{128}\left[1 + \frac{109}{10} + \frac{291}{40.4} + \frac{65}{40.4.16} + \frac{1525}{40.4.16.64} + \frac{5355}{40.4.16.64.8} + \frac{5313}{40.4.16.64.8.2} + \ldots\right]$$

Accordingly, if we subtract the logarithm of 128 from that of 175 we have the number 0.13582808 which stands first in the line of the third table with $b_4^{(3)}$. If the logarithm of $\frac{109}{40}$ is added to this we have 0.57119459 standing second and so on.

With regard to the last table of this article very slight modifications have been made in Walker's form of it, in order that those who have been accustomed to use the former edition may have no difficulty in using this. The first column of the table contains the designation of the function $b_n^{(j)} = \frac{1}{n!}\alpha^n D^n b^{(j)}$. The second the power of α which is a factor of the whole series. The factor which multiplies each numerical coefficient is put at the head of the column in each division of the table precisely in Walker's fashion. In the first six columns Walker's five decimals have been increased to eight decreasing by a unit each time till in the ninth column the number is the same as in the original edition. In this j was limited to 9, this contains three more. The original edition contained fourteen columns. This is an unnecessary extension and I have suppressed the last five modifying the form of the general ninth term so that the remainder of the series may be approximately taken into account as a geometrical progression. The disposable constants in the form of the term have been given such values as may lead to the closest representation of the remainder of the series between $\alpha = 0.55$ and $\alpha_t = 0.75$; the term is insignificant when $\alpha < 0.55$. In the first three divisions of the table it has been supposed sufficiently accurate to assign the exponent 2 to α in the denominator. But in the following divisions a greater degree of precision is attained by making the exponent have a somewhat different value. In one line of the table, that pertaining to $b_5^{(7)}$, it was found impossible to follow this method of accounting for the remainder of the series, and a couple of terms involving powers of $(\alpha - 0.65)$ was substituted. Walker's table loses nothing of precision by this cutting off of five terms at the end. The degree of precision attainable by the use of the formulæ of this table is indicated by saying that error of each function should not exceed a ten-millionth part provided α does not exceed 0.7.

It remains to say that negative characteristics have been avoided by the addition of ten. No confusion need result from this, the user of the table has only to remind himself that every characteristic greater than five has been augmented by ten.

The $b^{(i)}$ of larger subscripts than ½ have not been treated in this article for two reasons. First, their use in astronomy is not imperative, they can be avoided by a suitable arrangement of our formulæ. Second, the labor involved is a very serious matter. This improvement of WALKER table has cost as much labor as would be necessary to construct anew a table of logarithms of numbers to seven decimals.

	$b_4^{(3)}$		$b_4^{(4)}$		$b_4^{(5)}$		$b_4^{(6)}$		$b_4^{(7)}$		$b_4^{(8)}$		$b_4^{(9)}$		$b_4^{(10)}$	
	N.	D.	N.	D.	N.	D.	N.	D.	N.	D.	N.	D.	N.	D.	N.	D.
1	109	40	23	4	53	24	7	6	11	16	5	12	41	168	1	8
2	291	4	293	8	1261	20	475	64	677	56	7337	1120	1187	16	493	160
3	65	16	583	4	102393	224	5757	56	1745	48	−3253	4	−54347	176	−6262	33
4	1525	64	1709	256	661819	96	191929	96	882029	176	3831347	528	11042999	156	2837909	416
5	5355	8	2943	4	798127	8	222547	11	340395	4	18752989	52	8261333	8	4244139	16
6	5313	2	5775	4	605319	2	2046057	32	9029329	104	292569201	64	63310527	32	129210067	128
7	....	..	12441	4	4840979	16	13858247	16	109452103	32	419756911	4	202355035	64	116704415	34

	$b_4^{(11)}$		$b_4^{(12)}$		$b_5^{(0)}$		$b_5^{(1)}$		$b_5^{(2)}$		$b_5^{(3)}$		$b_5^{(4)}$		$b_5^{(5)}$	
	N.	D.	N.	D.	N.	D.	N.	D.	N.	D.	N.	D.	N.	D.	N.	D.
1	11	288	−1	36	151	48	197	16	4	1	487	40	53	12	69	8
2	8945	88	529	44	3047	200	2189	8	209	64	4071	4	585	8	703	4
3	−112533	26	−107871	364	54117	504	1181	40	21	10	37703	16	16773	64	22967	32
4	1806071	16	105994589	1024	131103	16	2991	16	1315	256	17885	64	1635	16	268553	32
5	43291957	256	676600655	68	1059975	32	24735	32	435	1	46705	8	1271	2	11063	8
6	1397300715	136	1284510015	8	14167125	4	.....	..	11095	64	36939	2	4543	8	9779	64
7	1919149785	4	33436713915	76	....	...	.....	..	.....	...	.....	..	.....	..	76791	16

	$b_5^{(6)}$		$b_5^{(7)}$		$b_5^{(8)}$		$b_5^{(9)}$		$b_5^{(10)}$		$b_5^{(11)}$		$b_5^{(12)}$	
	N.	D.	N.	D.	N.	D.	N.	D.	N.	D.	N.	D.	N.	D.
1	10	3	29	16	9	8	89	120	1	2	75	224	31	144
2	2321	128	343	8	3379	280	1901	16	1557	384	1033	8	1543	22
3	25707	14	23409	112	71779	48	8515	16	749	11	−577	22	−25347	104
4	3170333	384	122831	16	8058873	264	832241	132	629693	416	1409639	208	7252091	128
5	−775937	11	−684357	44	267189165	104	−45399995	104	−852221	2	−277835107	256	−3248731759	544
6	79391	32	4618385	104	157502535	8	−122251521	32	−67340571	1024	−2937849087	136	−1015637575	4
7	1581905	2	17987105	32	301098339	32	−1364806271	64	−226471248	17	−2353231775	4	−31133244825	152

	$b_6^{(0)}$		$b_6^{(1)}$		$b_6^{(2)}$		$b_6^{(3)}$		$b_6^{(4)}$		$b_6^{(5)}$		$b_6^{(6)}$		$b_6^{(7)}$	
	N.	D.	N.	D.	N.	D.	N.	D.	N.	D.	N.	D.	N.	D.	N.	D.
1	279	16	109	20	18	1	37	6	69	4	89	14	12	1	37	8
2	21149	40	9007	80	3093	64	1977	32	1823	8	12399	64	5575	128	1493	16
3	42631	40	1777	28	21047	10	112375	112	138977	64	91227	8	17853	2	52221	28
4	19987	16	31341	128	27963	256	41845	96	899465	16	981585	32	1706427	128	2112281	64
5	83889	32	28725	4	6100	1	86193	7	17251	6	2667	1	490911	1	11457239	22
6	.....	..	.....	...	16275685	19008	......	...	48891	8	34643	32	37319	32	38844433	208
7	.....	..	.....	...	....	.....	......	...	......	..	......	..	33553	2	18959975	4

	$b_6^{(8)}$		$b_6^{(9)}$		$b_6^{(10)}$		$b_6^{(11)}$		$b_6^{(12)}$		$b_7^{(0)}$		$b_7^{(1)}$	
	N.	D.	N.	D.	N.	D.	N.	D.	N.	D.	N.	D.	N.	D.
1	61	24	129	80	11	10	131	168	9	16	143	20	487	20
2	4819	40	871	4	2183	128	6365	32	199	14	15171	80	26447	16
3	77103	16	133823	192	4091	3	32358	11	5413	88	75937	196	434113	20
4	1040021	24	20560293	286	821437	352	9387479	416	9202893	1664	242659	128	1419349	896
5	17977175	88	38321785	16	−189455	26	−52291267	32	−103118071	32	336699	12	692067	4
6	192857235	104	84197147	4	3218581763	1024	86916082873	1088	7963041703	68	......	...	2278235	16
7	468780491	32	2984526403	512	3710575087	17	11736633821	2	81732669525	152	......	...	....	...

	$b_7^{(2)}$		$b_7^{(3)}$		$b_7^{(4)}$		$b_7^{(5)}$	
	N.	*D.*	*N.*	*D.*	*N.*	*D.*	*N.*	*D.*
1	257	32	19	2	241	28	323	14
2	10371	20	19357	96	80329	128	96171	64
3	386191	64	428695	16	334455	4	1432125	8
4	43213	12	6182505	32	2951545	21	38026905	32
5	544385	64	260978	21	215677	44	11808357	1
6	590835	4	4174351	64	8984283	128	535723	32
7	. . .	. .	5061309	8	. . .	. .	3217207	16

	$b_7^{(6)}$		$b_7^{(7)}$		$b_7^{(8)}$		$b_7^{(9)}$	
	N.	*D.*	*N.*	*D.*	*N.*	*D.*	*N.*	*D.*
1	547	64	127	8	73	12	161	48
2	6693	4	9833	16	19337	80	8161	20
3	307047	32	88053	4	134931	4	735933	64
4	422565	2	6535383	64	12601993	96	48395477	78
5	18169005	160	23015511	6	3675569885	572	3787830085	176
6	271005	12	87219519	16	2513551569	16	3102402885	4
7	35545135	64	386679	4	−302648449	4	−78589868941	512

	$b_7^{(10)}$		$b_7^{(11)}$		$b_7^{(12)}$	
	N.	*D.*	*N.*	*D.*	*N.*	*D.*
1	43	20	179	120	13	12
2	2887	32	10579	32	1247	56
3	20051	6	70548	7	42507	44
4	5344375	352	17671127	352	100449179	3328
5	105460625	52	2219624275	416	80939789	4
6	3073080373	128	46756388817	64	29595202359	544
7	−76036613509	136	−1938494328393	102	4220730422727	228

It is necessary to have the value of the numerical factor of the first significant term of LEVERRIER's $b_n^j = \frac{a^n}{n!} D^n b^{(j)}$. This is given in the following table. Two dots between two odd or two even numbers signify the product of all the odd or even integers between the limiting numbers.

j	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
0	2	1	$\frac{1}{2}$	$\frac{3.3}{2^5}$	$\frac{3.3}{2^4}$	$\frac{3.5.5}{2^6}$	$\frac{5.5}{2^8}$	$\frac{5^2.7^2}{2^{10}}$
1	1	1	$\frac{3.3}{2^5}$	$\frac{3}{2^3}$	$\frac{3.5.5}{2^6}$	$\frac{3.5}{2^6}$	$\frac{5^2.7^2}{2^{10}}$	$\frac{5^2.7}{2^{10}}$
2	$\frac{3}{4}$	$\frac{3}{2}$	$\frac{3}{2^2}$	$\frac{5}{4}$	$\frac{5}{2^4}$	$\frac{1}{2^8}\,5..9$	$\frac{3.5.7}{2^9}$	$\frac{5.7.9}{2^8}$
3	$\frac{3.5}{4.6}$	$\frac{1}{2}\,\frac{3.5}{4}$	$\frac{3.5}{2^3}$	$\frac{5}{2^5}$	$\frac{5.5.7}{2^7}$	$\frac{5.7}{2^7}$	$\frac{3.7^2.9}{2^{10}}$	$\frac{3.7.9}{2^{10}}$
4	$\frac{3..7}{4..8}$	$\frac{1}{2}\,\frac{3..7}{4.6}$	$\frac{1}{2^3}\,\frac{3..7}{4}$	$\frac{1}{2^4}\,5.7$	$\frac{5.7}{2^6}$	$\frac{3.7.9}{2^7}$	$\frac{7.9}{2^8}$	$\frac{7.9.11}{2^9}$
5	$\frac{3..9}{4..10}$	$\frac{1}{2}\,\frac{3..9}{4..8}$	$\frac{1}{2^3}\,\frac{3..9}{4.6}$	$\frac{1}{2^4}\,\frac{5..9}{4}$	$\frac{1}{2^7}\,.5.7.9$	$\frac{7.9}{2^7}$	$\frac{3.7^2.11}{2^{10}}$	$\frac{3.7.11}{2^{10}}$
6	$\frac{3..11}{4..12}$	$\frac{1}{2}\,\frac{3..11}{4..10}$	$\frac{1}{2^3}\,\frac{3..11}{4..8}$	$\frac{1}{2^4}\,\frac{5..11}{4.6}$	$\frac{1}{2^7}\,\frac{5..11}{4}$	$\frac{1}{2^8}\,7..11$	$\frac{3.7.11}{2^9}$	$\frac{3.11.13}{2^8}$
7	$\frac{3..13}{4..14}$	$\frac{1}{2}\,\frac{3..13}{4..12}$	$\frac{1}{2^3}\,\frac{3..13}{4..10}$	$\frac{1}{2^4}\,\frac{5..13}{4..8}$	$\frac{1}{2^7}\,\frac{5..13}{4.6}$	$\frac{1}{2^8}\,\frac{7..13}{4}$	$\frac{1}{2^{10}}\,.7.11.13$	$\frac{3.11.13}{2^{10}}$
8	$\frac{3..15}{4..16}$	$\frac{1}{2}\,\frac{3..15}{4..14}$	$\frac{1}{2^3}\,\frac{3..15}{4..12}$	$\frac{1}{2^4}\,\frac{5..15}{4..10}$	$\frac{1}{2^7}\,\frac{5..15}{4..8}$	$\frac{1}{2^8}\,\frac{7..15}{4.6}$	$\frac{1}{2^{10}}\,\frac{3.7..15.}{4}$	$\frac{1}{2^{11}}\,.3.11..15$
9	$\frac{3..17}{4..18}$	$\frac{1}{2}\,\frac{3..17}{4..16}$	$\frac{1}{2^3}\,\frac{3..17}{4..14}$	$\frac{1}{2^4}\,\frac{5..17}{4..12}$	$\frac{1}{2^7}\,\frac{5..17}{4..10}$	$\frac{1}{2^8}\,\frac{7..17}{4..8}$	$\frac{1}{2^{10}}\,\frac{3.7..17}{4.6}$	$\frac{1}{2^{11}}\,.3.11..17$
10	$\frac{3..19}{4..20}$	$\frac{1}{2}\,\frac{3..19}{4..18}$	$\frac{1}{2^3}\,\frac{3..19}{4..16}$	$\frac{1}{2^4}\,\frac{5..19}{4..14}$	$\frac{1}{2^7}\,\frac{5..19}{4..12}$	$\frac{1}{2^8}\,\frac{7..19}{4..10}$	$\frac{1}{2^{10}}\,\frac{3.7.11..19}{4..8}$	$\frac{1}{2^{11}}\,.\,\frac{3.11..19}{4.6}$
11	$\frac{3..21}{4..22}$	$\frac{1}{2}\,\frac{3..21}{4..20}$	$\frac{1}{2^3}\,\frac{3..21}{4..18}$	$\frac{1}{2^4}\,\frac{5..21}{4..16}$	$\frac{1}{2^7}\,\frac{5..21}{4..14}$	$\frac{1}{2^8}\,\frac{7..21}{4..12}$	$\frac{1}{2^{10}}\,\frac{3.7.11..21}{4..10}$	$\frac{1}{2^{11}}\,.\,\frac{3.11..21}{4..8}$
12	$\frac{3..23}{4..24}$	$\frac{1}{2}\,\frac{3..23}{4..22}$	$\frac{1}{2^3}\,\frac{3..23}{4..20}$	$\frac{1}{2^4}\,\frac{5..23}{4..18}$	$\frac{1}{2^7}\,\frac{5..23}{4..16}$	$\frac{1}{2^8}\,\frac{7..23}{4..14}$	$\frac{1}{2^{10}}\,\frac{3.7.11..23}{4..12}$	$\frac{1}{2^{11}}\,\frac{3.11..23}{4..10}$

Coefficients of the Perturbative Function.

Logarithm of the Coefficient of

Function	Factor		α^2	α^4	α^6	α^8	α^{10}	α^{12}	α^{14}	α^{16}
$b^{(0)}$	..	0.30103000	9.69897000	9.44909253	9.29073004	9.17474615	9.08323116	9.0076540	8.943285	$8.88723[1-9.95214\alpha^2]^{-1}$
$b^{(1)}$	α	0.00000000	9.57403127	9.36991129	9.23273809	9.12898865	9.04544260	8.9754694	8.915256	$8.86240[1-9.95459\alpha^2]^{-1}$
$b^{(2)}$	α^2	9.87506126	9.49485002	9.31191934	9.18698060	9.09120009	9.01325792	8.9474406	8.890432	$8.84013[1-9.95654\alpha^2]^{-1}$
$b^{(3)}$	α^3	9.79588002	9.43685807	9.26616185	9.14919204	9.05901541	8.98522920	8.9226171	8.868156	$8.81992[1-9.95818\alpha^2]^{-1}$
$b^{(4)}$	α^4	9.73788807	9.39110058	9.22837329	9.11700736	9.03098669	8.96040561	8.9003407	8.847953	$8.80144[1-9.95957\alpha^2]^{-1}$
$b^{(5)}$	α^5	9.69213058	9.35331202	9.19618860	9.08897863	9.00616310	8.93812922	8.8801373	8.829469	$8.78441[1-9.96073\alpha^2]^{-1}$
$b^{(6)}$	α^6	9.65434202	9.32112734	9.16815988	9.06415505	8.98388671	8.91792583	8.8616539	8.812436	$8.76861[1-9.96173\alpha^2]^{-1}$
$b^{(7)}$	α^7	9.62215734	9.29309862	9.14333630	9.04187866	8.96368332	8.89944243	8.8446205	8.796642	$8.75389[1-9.96262\alpha^2]^{-1}$
$b^{(8)}$	α^8	9.59412861	9.26827503	9.12105990	9.02167527	8.94519992	8.88240909	8.8288263	8.781918	$8.74010[1-9.96340\alpha^2]^{-1}$
$b^{(9)}$	α^9	9.56930503	9.24599864	9.10085652	9.00319186	8.92816658	8.86661482	8.8141030	8.768130	$8.72714[1-9.96410\alpha^2]^{-1}$
$b^{(10)}$	α^{10}	9.54702863	9.22579525	9.08237311	8.98615852	8.91237231	8.85189156	8.8003147	8.755165	$8.71490[1-9.96474\alpha^2]^{-1}$
$b^{(11)}$	α^{11}	9.52682525	9.20731185	9.06533977	8.97036426	8.89764905	8.83810328	8.7873497	8.742931	$8.70332[1-9.96529\alpha^2]^{-1}$
$b^{(12)}$	α^{12}	9.50834184	9.19027851	9.04954550	8.95564100	8.88386077	8.82513830	8.7751153	8.731349	$8.69232[1-9.96578\alpha^2]^{-1}$

Function	Factor	β^2	$\alpha^2\beta^2$	$\alpha^4\beta^2$	$\alpha^6\beta^2$	$\alpha^8\beta^2$	$\alpha^{10}\beta^2$	$\alpha^{12}\beta^2$	$\alpha^{14}\beta^2$	$\alpha^{16}\beta^2$
$b_1^{(0)}$	α^{-2}	....	0.00000000	9.09691001	8.67094128	8.38764005	8.17474615	8.0040499	7.861526	$7.73916[1-9.89641\alpha^2]^{-1}$
$b_1^{(1)}$	α^{-1}	0.00000000	n9.09691001	8.67094128	8.38764005	8.17474615	8.00404992	7.8615260	7.739165	$7.63195[1-9.90770\alpha^2]^{-1}$
$b_1^{(2)}$	..	0.17609126	n9.39794001	n8.29073004	....	7.50573936	7.54713205	7.5081079	7.449130	$7.38528[1-9.93718\alpha^2]^{-1}$
$b_1^{(3)}$	α	0.27300127	n9.70570339	n8.87619077	n8.36307186	n7.94507206	n7.55386543	n7.1278967	n6.475459	6.32160 ......
$b_1^{(4)}$	α^2	0.33994806	n9.85183142	n9.09007059	n8.64651668	n8.31082738	n8.03098669	n7.7843144	n7.557918	$n7.34280[1-9.78645\alpha^2]^{-1}$
$b_1^{(5)}$	α^3	0.39110058	n9.94538779	n9.21790785	n8.80498198	n8.49881787	n8.24920115	n8.0350392	n7.845160	$n7.67277[1-9.84038\alpha^2]^{-1}$
$b_1^{(6)}$	α^4	0.43249327	n0.01336396	n9.30733906	n8.91288737	n8.62346306	n8.38965205	n8.1909271	n8.016556	$n7.86013[1-9.86046\alpha^2]^{-1}$
$b_1^{(7)}$	α^5	0.46725538	n0.06639736	n9.37528537	n8.99357398	n8.71544408	n8.49195710	n8.3029149	n8.137825	$n7.99046[1-9.86954\alpha^2]^{-1}$
$b_1^{(8)}$	α^6	0.49721860	n0.10970812	n9.42964303	n9.05739082	n8.78759206	n8.57161925	n8.3894936	n8.230883	$n8.08968[1-9.87544\alpha^2]^{-1}$
$b_1^{(9)}$	α^7	0.52354754	n0.14622198	n9.47470223	n9.10981136	n8.84649653	n8.63634458	n8.4595314	n8.305840	$n8.16925[1-9.87991\alpha^2]^{-1}$
$b_1^{(10)}$	α^8	0.54702863	n0.17773381	n9.51303696	n9.15406633	n8.89598189	n8.69052356	n8.5179803	n8.368220	$n8.23529[1-9.88345\alpha^2]^{-1}$
$b_1^{(11)}$	α^9	0.56821793	n0.20541949	n9.54630436	n9.19221301	n8.93845788	n8.73689135	n8.5678872	n8.421379	$n8.29147[1-9.88624\alpha^2]^{-1}$
$b_1^{(12)}$	α^{10}	0.58752309	n0.23008906	n9.57562939	n9.22563676	n8.975535583	n8.77726349	n8.6112585	n8.467505	$n8.34014[1-9.88835\alpha^2]^{-1}$

Function	Factor	β^2	β^4	$\alpha^2\beta^4$	$\alpha^4\beta^4$	$\alpha^6\beta^4$	$\alpha^8\beta^4$	$\alpha^{10}\beta^4$	$\alpha^{12}\beta^4$	$\alpha^{14}\beta^4$
$b_2^{(0)}$	α^{-2}		9.69897000	9.83727270	8.73788807	8.18003174	7.79630393	7.5021051	7.263040	$7.06151[1-9.83130\alpha^2]^{-1}$
$b_2^{(1)}$	α^{-1}	...	0.05115252	8.97197128	8.42106381	8.04085257	7.74877741	7.5111344	7.310619	$7.13710[1-9.85240\alpha^2]^{-1}$
$b_2^{(2)}$	..	9.87506126	0.05115252	8.88179465	8.46682130	8.15895188	7.91227954	7.7059184	7.528311	$7.37231[1-9.86486\alpha^2]^{-1}$
$b_2^{(3)}$	α	0.27300127	9.93418272	9.45059789	8.76421701	8.40060902	8.13889208	7.9295290	7.753111	$7.59981[1-9.86810\alpha^2]^{-1}$
$b_2^{(4)}$	α^2	0.51603932	9.61294933	9.80328103	9.03428880	8.62921182	8.34725665	8.1278967	7.946617	$7.79115[1-9.86757\alpha^2]^{-1}$
$b_2^{(5)}$	α^3	0.69213058	n9.26616185	0.04258628	9.24641215	8.82002991	8.52417909	8.2961977	8.109673	$7.95108[1-9.86582\alpha^2]^{-1}$
$b_2^{(6)}$	α^4	0.83043328	n9.95537201	0.22110254	9.41531449	8.97808572	8.67355540	8.4392940	8.218416	$8.08688[1-9.86427\alpha^2]^{-1}$
$b_2^{(7)}$	α^5	0.94437663	n0.23758129	0.36344438	9.55386543	9.11100923	8.80111714	8.5624686	8.368232	$8.20422[1-9.86260\alpha^2]^{-1}$
$b_2^{(8)}$	α^6	1.04128664	n0.42343238	0.48097171	9.67058528	9.22482476	8.91161636	8.6699370	8.473175	$8.30716[1-9.86113\alpha^2]^{-1}$
$b_2^{(9)}$	α^7	1.12560753	n0.56384213	0.58106284	9.77106186	9.32388683	9.00863845	8.7648749	8.566236	$8.39863[1-9.85992\alpha^2]^{-1}$
$b_2^{(10)}$	α^8	1.20024115	n0.67736240	0.66814901	9.85906944	9.41132579	9.09485037	8.8496629	8.649638	$8.48079[1-9.85900\alpha^2]^{-1}$
$b_2^{(11)}$	α^9	1.26718794	n0.77295437	0.74517830	9.93724614	9.48942620	9.17225190	8.9261048	8.725064	$8.55525[1-9.85835\alpha^2]^{-1}$
$b_2^{(12)}$	α^{10}	1.32788578	n0.85567186	0.81420677	0.00749552	9.55988917	9.24236477	8.9955877	8.793810	$8.62325[1-9.85792\alpha^2]^{-1}$

Function	Factor	β^2	β^4	β^6	$\alpha^2\beta^6$	$\alpha^4\beta^6$	$\alpha^6\beta^6$	$\alpha^8\beta^6$	$\alpha^{10}\beta^6$	$\alpha^{12}\beta^6$
$b_3^{(0)}$	α^{-2}	...	...	0.05115252	9.72529894	8.48106174	7.81900382	7.3523121	6.988851	$6.69023[1-9.76828\alpha^{2.06}]^{-1}$
$b_3^{(1)}$	α^{-1}	....	9.57403127	0.20242020	8.87619077	8.22648914	7.76615151	7.4065615	7.110547	$6.85861[1-9.79885\alpha^{2.05}]^{-1}$
$b_3^{(2)}$	..	....	0.09691001	0.20454389	8.76785129	8.27694175	7.90367937	7.5999265	7.343012	$7.12007[1-9.81905\alpha^{2.044}]^{-1}$
$b_3^{(3)}$	α	0.79588002	0.32415379	0.20849804	8.61808897	8.23947147	7.93255550	7.6735527	7.449130	$7.25095[1-9.83677\alpha^{2.04}]^{-1}$
$b_3^{(4)}$	α^2	0.33994806	0.43685807	0.25967972	n9.13114982	n7.37139316	7.58249030	7.5216807	7.383169	$7.23566[1-9.86405\alpha^{2.02}]^{-1}$
$b_3^{(5)}$	α^3	0.69213058	0.47328734	0.36691520	n9.75886093	n8.71324310	n7.96824181	n7.1481001	6.627307	$6.90555[1-9.94797\alpha^{1.88}]^{-1}$
$b_3^{(6)}$	α^4	0.95537201	0.43249327	0.51010988	n0.13116696	n9.14641227	n8.52470720	n8.0228302	n7.566066	$n7.09796[1-8.9905\alpha^{0.5}]^{-1}$
$b_3^{(7)}$	α^5	1.16622538	0.26313539	0.66493582	n0.40413231	n9.44053009	n8.85176444	n8.3984586	n8.015614	$n7.67302[1-9.67860\alpha^2]^{-1}$
$b_3^{(8)}$	α^6	1.34231664	9.41803735	0.81651989	n0.62117304	n9.66824604	n9.09429476	n8.6603355	n8.302194	$n7.99132[1-9.73288\alpha^{\frac{61}{30}}]^{-1}$
$b_3^{(9)}$	α^7	1.49358431	n0.31942756	0.95858189	n0.80174115	n9.85540774	n9.28983694	n8.8661678	n8.520319	$n8.22395[1-9.75366\alpha^{2.044}]^{-1}$
$b_3^{(10)}$	α^8	1.62620988	n0.72311989	1.08943670	n0.95647698	n0.01475099	n9.15161663	n9.0373299	n8.698823	$n8.41078[1-9.76483\alpha^{2.05}]^{-1}$
$b_3^{(11)}$	α^9	1.74430919	n0.97333653	1.20936046	n1.09190196	n0.15365994	n9.59738944	n9.1844702	n8.850854	$n8.56821[1-9.77105\alpha^{2.05}]^{-1}$
$b_3^{(12)}$	α^{10}	1.85076452	n1.16155436	1.31932357	n1.21232373	n0.27685754	n9.72351520	n9.3137975	n8.983685	$n8.70485[1-9.77581\alpha^{2.052}]^{-1}$

Coefficients of the Perturbative Function.—*Continued.*

Logarithm of the Coefficient of

Function	Factor	β^2	β^4	β^6	β^8	$\alpha^2\beta^8$	$\alpha^4\beta^8$	$\alpha^6\beta^8$	$\alpha^8\beta^8$	$\alpha^{10}\beta^8$
$b_4^{(0)}$	α^{-2}			9.44909253	0.31930129	9.63976160	8.28112440	7.5335257	6.996612	$6.57323[1-9.67535\alpha^{2.07}]^{-1}$
$b_4^{(1)}$	α^{-1}			0.06888129	0.39196143	8.79841729	8.06718150	7.5387081	7.120409	$6.77276[1-9.72402\alpha^{2.06}]^{-1}$
$b_4^{(2)}$			9.49485002	0.38937376	0.39367801	8.67906441	8.12366286	7.6935464	7.339295	$7.03705[1-9.75466\alpha^{2.05}]^{-1}$
$b_4^{(3)}$	α		0.13582808	0.57119459	0.39560109	8.54050147	8.10167798	7.7470776	7.442628	$7.17703[1-9.78080\alpha^{2.045}]^{-1}$
$b_4^{(4)}$	α^2	9.73788807	0.49755592	0.69960571	0.39634666	8.15518020	8.08916841	7.7798701	7.511113	$7.27309[1-9.80074\alpha^{2.04}]^{-1}$
$b_4^{(5)}$	α^3	0.39110058	0.73516521	0.81057443	0.36988160	9.19807933	8.37632211	7.9552044	7.654033	$7.40834[1-9.80507\alpha^{2.05}]^{-1}$
$b_4^{(6)}$	α^4	0.83013328	0.89738007	0.92279566	0.25811026	9.79878339	8.82167185	8.2800179	7.906688	$7.62047[1-9.78868\alpha^{2.067}]^{-1}$
$b_4^{(7)}$	α^5	1.16622538	1.00349808	1.04450604	9.77147156	0.23264633	9.21708665	8.6237256	8.202144	$7.87784[1-9.76535\alpha^{2.075}]^{-1}$
$b_4^{(8)}$	α^6	1.43922665	1.05901541	1.17634590	n0.22105146	0.56918196	9.54320062	8.9301788	8.484888	$8.13714[1-9.74501\alpha^{2.075}]^{-1}$
$b_4^{(9)}$	α^7	1.66967557	1.05715015	1.31469703	n0.72990922	0.84469608	9.81556918	9.1948450	8.738401	$8.37784[1-9.73202\alpha^{2.075}]^{-1}$
$b_4^{(10)}$	α^8	1.86924793	0.96615794	1.45488488	n1.04023708	1.07742931	0.04810030	9.4243971	8.962652	$8.59521[1-9.72391\alpha^{2.075}]^{-1}$
$b_4^{(11)}$	α^9	2.04533919	0.62733938	1.59301137	n1.27777058	1.27910552	0.25053795	9.6258817	9.161641	$8.79049[1-9.71917\alpha^{2.075}]^{-1}$
$b_4^{(12)}$	α^{10}	2.20294701	n0.61664154	1.72664754	n1.17499518	1.15707422	0.42961102	9.8049291	9.339602	$8.96647[1-9.71474\alpha^{2.072}]^{-1}$

Function	Factor	β^2	β^4	β^6	β^8	β^{10}	$\alpha^2\beta^{10}$	$\alpha^4\beta^{10}$	$\alpha^6\beta^{10}$	$\alpha^8\beta^{10}$
$b_5^{(0)}$	α^{-2}			......	0.06888129	0.56661700	9.57048251	8.1175132	7.297672	$6.70021[1-9.55378\alpha^{2.08}]^{-1}$
$b_5^{(1)}$	α^{-1}			9.36991129	0.46025753	0.60294708	8.73289122	7.9323377	7.344683	$6.87452[1-9.63180\alpha^{2.07}]^{-1}$
$b_5^{(2)}$				0.09007059	0.69213058	0.60403690	8.60610991	7.9945761	7.514110	$7.11460[1-9.67878\alpha^{2.06}]^{-1}$
$b_5^{(3)}$	α		9.43685807	0.52232704	0.84243919	0.60199402	8.47492708	7.9887115	7.585803	$7.24026[1-9.71563\alpha^{2.05}]^{-1}$
$b_5^{(4)}$	α^2		0.16925183	0.81434646	0.95413617	0.60541137	8.39019840	7.9797962	7.629903	$7.32388[1-9.74528\alpha^{2.05}]^{-1}$
$b_5^{(5)}$	α^3	9.69213058	0.62788968	1.03393593	1.04293489	0.60571065	8.31746357	7.9628551	7.653751	$7.37916[1-9.76755\alpha^{2.04}]^{-1}$
$b_5^{(6)}$	α^4	0.43249327	0.95537201	1.21383719	1.11208541	0.61880767	n8.96613656	6.1709311	7.469311	$7.32577[1-9.81117\alpha^2]^{-1}$
$b_5^{(7)}$	α^5	0.91437663	1.20265465	1.37246078	1.15733150	0.67310128	n9.77565869	n8.5878328	n7.673154	$[6.68532+2.0289(\alpha-.65)+0.1050(\alpha-.65)^2$
$b_5^{(8)}$	α^6	1.34231664	1.39346916	1.52085681	1.16682499	0.79549777	n0.29900896	n9.1663877	n8.419780	$n7.79447[1-9.30169\alpha^{1.74}]^{-1}$
$b_5^{(9)}$	α^7	1.66967557	1.53988433	1.66535646	1.11243901	0.98192954	n0.70170290	n9.5832857	n8.868388	$n8.30542[1-9.52323\alpha^{2.02}]^{-1}$
$b_5^{(10)}$	α^8	1.94842917	1.64739918	1.80892315	0.89615709	1.20171080	n1.03210417	n9.9195288	n9.215816	$n8.67189[1-9.56437\alpha^{2.05}]^{-1}$
$b_5^{(11)}$	α^9	2.19146722	1.71628047	1.95222954	n0.35688235	1.42675111	n1.31319036	n0.2038938	n9.505469	$n8.96981[1-9.57995\alpha^{2.06}]^{-1}$
$b_5^{(12)}$	α^{10}	2.40706702	1.74006623	2.09464778	n1.29317508	1.64250179	n1.55815349	n0.1511184	n9.755760	$n9.22448[1-9.58912\alpha^{2\frac{1}{4}}]^{-1}$

Function	Factor	β^2	β^4	β^6	β^8	β^{10}	β^{12}	$\alpha^2\beta^{12}$	$\alpha^4\beta^{12}$	$\alpha^6\beta^{12}$
$b_6^{(0)}$	α^{-2}				9.29073004	0.53221126	0.80983990	9.5122156	7.979118	$7.09693[1-9.38258\alpha^{2.09}]^{-1}$
$b_6^{(1)}$	α^{-1}				0.07783613	0.81423263	0.82829631	8.6762455	7.815461	$7.17555[1-9.51268\alpha^{2.08}]^{-1}$
$b_6^{(2)}$				9.31191934	0.56719184	0.99611928	0.82892957	8.5440832	7.882829	$7.35726[1-9.58418\alpha^{2\frac{1}{15}}]^{-1}$
$b_6^{(3)}$	α			0.11125989	0.90131036	1.12396533	0.82941034	8.4181130	7.886843	$7.14294[1-9.63878\alpha^{2.06}]^{-1}$
$b_6^{(4)}$	α^2		9.39110058	0.62788968	1.14673727	1.22271356	0.82963492	8.3343137	7.883638	$7.49662[1-9.67840\alpha^{2.05}]^{-1}$
$b_6^{(5)}$	α^3		0.19841006	1.00167203	1.33918871	1.30313546	0.82979000	8.2638851	7.872328	$7.52902[1-9.71151\alpha^{2.05}]^{-1}$
$b_6^{(6)}$	α^4	9.65434202	0.73352326	1.29337692	1.49781326	1.37097980	0.82989485	8.2056721	7.858413	$7.54941[1-9.73598\alpha^{2.04}]^{-1}$
$b_6^{(7)}$	α^5	0.46725538	1.13236711	1.53410521	1.63073256	1.43145909	0.82336469	9.0355502	8.121999	$7.67688[1-9.71666\alpha^{2.08}]^{-1}$
$b_6^{(8)}$	α^6	1.01128664	1.44640524	1.74197234	1.74196670	1.49172629	0.78492296	9.7984041	8.678988	$8.03915[1-9.61617\alpha^{2\frac{1}{4}}]^{-1}$
$b_6^{(9)}$	α^7	1.49358131	1.70108401	1.92845249	1.83166387	1.56179637	0.62809282	0.3678845	9.208193	$8.49319[1-9.52803\alpha^{2.12}]^{-1}$
$b_6^{(10)}$	α^8	1.86924793	1.91061061	2.10109169	1.89674618	1.65294830	n9.60090675	0.8207653	9.652093	$8.91450[1-9.48548\alpha^{2.1}]^{-1}$
$b_6^{(11)}$	α^9	2.19146722	2.08342924	2.26480637	1.92959695	1.77307093	n1.01380113	1.1978432	0.027257	$9.28259[1-9.46840\alpha^{2.09}]^{-1}$
$b_6^{(12)}$	α^{10}	2.47401381	2.22413634	2.12261887	1.91272115	1.92205117	n1.16631460	1.5215499	0.351023	$9.60118[1-9.46231\alpha^{2.09}]^{-1}$

Function	Factor	β^2	β^4	β^6	β^8	β^{10}	β^{12}	β^{14}	$\alpha^2\beta^{14}$	$\alpha^4\beta^{14}$
$b_7^{(0)}$	α^{-2}					0.07783613	0.93214217	1.0517301	9.461914	$7.85925[1-9.09759\alpha^{2.09}]^{-1}$
$b_7^{(1)}$	α^{-1}				9.23273809	0.61923706	1.14996453	1.0611609	8.626339	$7.71234[1-9.34379\alpha^{2.08}]^{-1}$
$b_7^{(2)}$					0.09007059	0.99485373	1.29971125	1.0645128	8.490076	$7.78419[1-9.16012\alpha^{2.07}]^{-1}$
$b_7^{(3)}$	α			9.26616185	0.65532793	1.26969867	1.41088905	1.0617551	8.367975	$7.79592[1-9.54165\alpha^{2.06}]^{-1}$
$b_7^{(4)}$	α^2			0.13146327	1.06632228	1.48196763	1.49937297	1.0648734	8.285175	$7.79764[1-9.59945\alpha^{2.055}]^{-1}$
$b_7^{(5)}$	α^3		9.35331202	0.71638651	1.38101815	1.65389195	1.57285503	1.0649535	8.216554	$7.79097[1-9.61361\alpha^{2.05}]^{-1}$
$b_7^{(6)}$	α^4		0.22421733	1.15602468	1.64159849	1.79803224	1.63569091	1.0650085	8.159473	$7.78080[1-9.67916\alpha^{2.05}]^{-1}$
$b_7^{(7)}$	α^5	9.62215734	0.82287107	1.50763341	1.85763153	1.92197846	1.69057677	1.0650498	8.109727	$7.76823[1-9.70628\alpha^{2.04}]^{-1}$
$b_7^{(8)}$	α^6	0.49721860	1.28136021	1.80133646	2.04299911	2.03105538	1.73854480	1.0683579	n8.847979	$6.93130[1-9.99911\alpha^{1.2}]^{-1}$
$b_7^{(9)}$	α^7	1.12560753	1.65119217	2.05507967	2.20399460	2.12986649	1.77793954	1.0891873	n9.783586	$n8.48947[1-8.78317\alpha^{1.5}]^{-1}$
$b_7^{(10)}$	α^8	1.62620988	1.95861831	2.28017669	2.34101469	2.22323292	1.80212299	1.1596966	n0.419607	$n9.17628[1-9.17031\alpha^{2.03}]^{-1}$
$b_7^{(11)}$	α^9	2.01533919	2.21901097	2.18545258	2.46439463	2.31663151	1.79655339	1.3139319	n0.922958	$n9.68995[1-9.21551\alpha^{2.07}]^{-1}$
$b_7^{(12)}$	α^{10}	2.40706702	2.44182943	2.67556420	2.56470553	2.41600815	1.72016383	1.5476242	n1.343856	$n0.11422[1-9.22799\alpha^{2.08}]^{-1}$

SETH CARLO CHANDLER.

In the death of Doctor SETH CARLO CHANDLER on December 31, 1913, the *Astronomical Journal* lost not merely an associate editor, but the man who for years stood behind the journal, its editor, a heavy contributor of articles, and even backing it financially from his private purse. If the *Astronomical Journal* was the pet undertaking of its founder Dr. B. A. GOULD, it in no less measure became an object of absorbing interest to Doctor CHANDLER when he assumed the responsibilities of the editorship, upon the death of the founder on November 26, 1896. Until the year 1905 the *Astronomical Journal* flourished under the care of Doctor CHANDLER, but ill health overtook him at this point, and though at times he was able to perform his editorial duties there were periods of some duration when no copy of the *Journal* appeared. Finally in 1909 Doctor CHANDLER turned the *Journal* over to Professor LEWIS BOSS, but urged by the new editor consented to retain his connection with the *Journal* as an associate editor.

Doctor CHANDLER was born at Boston, Mass., September 17, 1846, the son of SETH CARLO and MARY (Cheever) CHANDLER. He was educated in the English High School at Boston, during his graduating year performing some computations for Prof. BENJAMIN PEIRCE. After graduation in 1861 he worked for some time with Dr. B. A. GOULD as private assistant, and joined the U. S. Coast Survey as aid in 1864, at a time when the longitude-determinations of the Coast Survey were being developed by Doctor GOULD. He was offered an opportunity to accompany Doctor GOULD on his expedition to Cordoba, Argentina, but declined the offer to accept a position as actuary with the Continental Life Insurance Co., taking up his residence at New York.

On October 20, 1870, Doctor CHANDLER married Miss CAROLINE M. HERMAN of Boston, who with five daughters survives him.

In 1877 he returned to Boston to accept a position as consulting actuary for the Union Mutual Life Insurance Co., of Boston, a position he held for several years.

It was about the time he moved to Cambridge in 1881 that he became associated with the Harvard College Observatory, and resumed his astronomical work. Realizing the necessity of the telegraphic transmission of certain astronomical observations and discoveries, Doctor CHANDLER with Mr. JOHN RITCHIE formulated a code which for many years distributed the desired information to observatories all over the United States. It was during his connection with Harvard College that he constructed the Almucanter, for which he received a medal from the Massachusetts Charitable Mechanics Association.

From 1886 on, Doctor CHANDLER became a private investigator. To enumerate the long list of his investigations is impossible in these pages, but their general character may be summed up. Among his earlier papers cometary articles occupy a prominent position. Perhaps the best known of these is his identification of Comet *d* 1889 with LEXELL'S comet of 1770.

He soon became interested in variable stars, their classification and the general laws pertaining to stellar variations, publishing at intervals catalogues of variable stars, and adding a great store of knowledge to this subject. An interesting paper on this subject was a treatment of *Algol* showing the possibility of a triple system, the binary system revolving about another invisible star with a period of about one hundred and thirty years.

Perhaps Doctor CHANDLER is best known for his discovery and subsequent elaboration of the variation of latitude. From the treatment of a long list of catalogues he established beyond a doubt the existence of a real periodic change in the position of the Earth's axis.

There were many other papers by Doctor CHANDLER dealing with a variety of subjects.

Some years previous to his death Doctor CHANDLER was employed upon the formation of a standard system of magnitudes derived from a systematic treatment of various catalogues. The discussion attempted to rid the

observations of all known sources of error. While the catalogue was never printed it is hoped that the labor was far enough advanced so that it may be completed and published.

Just previous to his death, Doctor CHANDLER had been working up another article dealing with latitude variation, and it is probable that the discussion was far enough advanced to be recovered and published.

In all, Doctor CHANDLER was the author of over two hundred astronomical papers. This fact will sufficiently account for a very busy life. A man possessed of great nervous energy, he was accustomed to work at forced speed whenever anything new or interesting came to his notice. Thus he would sit at his desk for hours on end, barely partaking of nourishment, in order to complete a comet orbit as soon as the necessary data were at hand.

But in spite of his great scientific activity he was always interested in outside affairs. Much of his time in later years was taken up with the management of the family estate.

Personally Doctor CHANDLER was an interesting and entertaining man, a good conversationalist with a store of information to draw upon, witty and with a keen sense of humor.

In 1904 he moved from Cambridge to Wellesley, Mass., and it was at the latter place that he succumbed rapidly to an attack of pneumonia after an illness lasting five days. Doctor CHANDLER had apparently so far improved in health during recent years that his sudden death came as a distinct shock. The *Astronomical Journal* has lost a friend who can not be replaced.

In 1895 Doctor CHANDLER received the WATSON Medal of the National Academy of Sciences, and in 1896 he received the gold medal of the Royal Astronomical Society. He received the degree of LL.D. from DePauw in 1891. He was a member of the National Academy of Sciences; F. A. A., vice president, 1886; Astronomical and Astrophysical Society of America; and foreign associate of the Royal Astronomical Society.

ASTEROID (475.) OCLLO,

CONSTANTS AND EPHEMERIS FOR ϑ OF DECEMBER, 1914,

BY F. E. SEAGRAVE.

$$\left.\begin{aligned} x &= r(9.99224)\ \sin(124^\circ\ 26'\ 20''.00 + u) \\ y &= r(9.89693)\ \sin(\ 42\ 59\ 26.02 + u) \\ z &= r(9.80796)\ \sin(\ 21\ 17\ 23.90 + u) \end{aligned}\right\}\ \text{Constants.}$$

$$\left.\begin{aligned} M &= \ 40^\circ\ 39'\ 27''.31 \\ \pi &= 337^\circ\ 23'\ 29''.04 \\ \Omega &= \ 35^\circ\ 53'\ 33''.00 \\ i &= \ 18^\circ\ 36'\ 42''.02 \\ \text{Log } e &= 9.580410 \\ \text{Log } a &= 0.414178 \\ \text{Log } q &= 0.20618. \\ \mu &= 848''.6731. \end{aligned}\right\}\ \text{Elements.}$$

1914	R.A. h m s	Dec. ° ′ ″	Log r	Log Δ
Oct 16	=5 19 6.	+33 37 40.	0.32076.	0.14593.
20	=5 18 18.	+34 28 43.	0.32434.	0.13941.
24	=5 16 47.	+35 19 14.	0.32792.	0.13351.
28	=5 14 29.	+36 8 49.	0.33148.	0.12826.
Nov. 1	=5 11 27.	+36 56 54.	0.33500.	0.12387.
5	=5 7 39.	+37 42 42.	0.33852.	0.12041.
9	=5 3 11.	+38 25 44.	0.34200.	0.11796.
13	=4 58 4.	+39 5 10.	0.34548.	0.11672.
17	=4 52 30.	+39 40 33.	0.34800.	0.11666.
21	=4 46 28.	+40 10 51.	0.35230.	0.11799.
25	=4 40 9.	+40 35 49.	0.35566.	0.12070.

ϑ = Dec. 1, 1914.

E and O = 1914 — Oct. 16.50 G.M.T.

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ACTING EDITOR, BENJAMIN BOSS, ALBANY, N.Y.; ASSOCIATE EDITORS, GEORGE W. HILL, DIRECTOR OF THE GOULD FUND OF THE NATIONAL ACADEMY OF SCIENCES AND PROF. ERNEST W. BROWN, OF YALE UNIVERSITY. PUBLISHED BY THE DUDLEY OBSERVATORY, ALBANY, N.Y., U.S.A., TO WHICH ALL COMMUNICATIONS SHOULD BE ADDRESSED PRICE, $5.00 THE VOLUME. PRESS OF THOS. P. NICHOLS & SON CO., LYNN, MASS. *Closed, Jan. 23, 1914*

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MEMOIR ON THE THEORY OF DETERMINING ORBITS,

By F. R. Moulton.

1. Introduction.

The theory of determining orbits has been the subject of such a multitude of papers that still another should not be added without good reasons.

One of the reasons for this paper is that an attempt is made to clarify the problem mathematically. In spite of the fact that it involves only the theory of the problem of two bodies and some of the fundamental principles of implicit functions, the complexity of its explicit development is so great that beginners often complain that its form is repulsive and that it is difficult to seize on its essentials. The solution of the problem depends upon two sets of conditions, the geometrical and the dynamical. They are very simple when they are imposed directly and separately, as they are here. The resulting equations contain all there is in the problem, if there are no disturbing forces, and they are subject to no artificial difficulties. If there are disturbing forces, as there might be exceptionally, they can be taken into account by adding simple terms to the same equations. With the perturbations taken into account the equations contain all that is in any theory of orbits — in fact, all there is in the problem.

The solution of the fundamental equations is a purely mathematical problem. The notations in general use are very poorly adapted to showing what its essentials are, and they are responsible for a large part of the difficulties. The equations are treated here so as to make all the operations in standard mathematical form. In particular, one of the important steps in finding the solution is simply a matter of determinants, and the use of this notation not only simplifies the work but leads to important practical advantages. Another mathematical theory which is involved is that of implicit functions, but heretofore it has been obscured by the many equations, the complex notation, and the so-called "successive hypotheses" in computing the final result. This part of the work is developed here so that its essentials are on the surface, and this leads to a solution in series in which no successive approximations are required; or in the terminology of Charlier*, to a "*direct analytical solution.*" This solution is not to be recommended in practice any more than a series solution of an algebraic equation would be recommended when the roots could be found more conveniently by Newton's or Horner's methods; but it is important in showing exactly what is involved and the limitations to which all methods are subject. It is only to be expected that an analytic solution exists, because all the functions involved are analytic functions.

A second and chief reason for publishing this paper is to show precisely where the results in orbit calculation become poorly determined, to define the extent of the indetermination, and to make use of the fact that the final results are only partially determinate to abbreviate the computation. This statement requires a little explanation. Everyone has noticed that if several orbits of a comet or small planet are published they generally differ considerably among themselves. This divergence of results is not to be explained by errors of the observations, or of the computations, but by the inherent partial indetermination of the problem. For example, three orbits of Comet *a* 1913 (Schaumasse), all computed by members of the astronomical department of the University of California, and published as Lick Observatory *Bulletins*, Nos. 227 and 228, gave results as follows:

	T	ω	Ω	i	q
I.	May 17.9103	57°27′.6	317°00′.0	153°33′.9	1.4400
II.	May 16.2792	54 35 .9	315 42 .8	152 43 .1	1.4528
III.	May 15.14920	53 1 18″.5	315 4 49″.0	152 21 7″.1	1.45790

* In an important memoir in *Meddelande från Lunds Astr. Obs.*, No. 46, Charlier shows how the method of Lagrange leads to a "direct analytical solution."

The first orbit was based on three observations which were very near together; the second, on one of the first three and on two others made at intervals of two days; and the third, from three averages of observations covering an interval of thirty-six days. It is very probable that the high order of excellence of the work for which the Berkeley computers have long been noted has been maintained in the present cases, and that the large discrepancies among the three sets of elements are due entirely to the partial indetermination which will be shown to be fundamental in the problem, and which no artifice can remedy.

The fact that the observations may be very closely represented by a set of elements is no guarantee of the exactness of the latter, as seems generally to have been supposed. The greatest difference between the computed and observed coördinates in the case of the second elements above is only 8″, while in the case of the third it is only 5″.7, yet the differences in the angular elements of the two sets are from 22′ to 1° 34′. The reason for this phenomenon is explained and illustrated in section 2 in a simpler problem which has intimate relations to that of determining orbits.

It is shown how to determine almost at the beginning of the computation the extent to which the final results will be indeterminate; and the work is so arranged that the number of places employed in all the calculations may be reduced correspondingly without impairing the accuracy. The practical computer will recognize at once what a great amount of labor will be saved if four- or five-place tables may be used in place of six- or seven-place tables. The reasons why so many places are now almost invariably used in the computations are that there has been no adequate theory of the degree of determination of the results to serve as a guide, and that all the treatises on orbits give *model computations* at least to six, and generally to seven, places even though the final results may not be correct beyond three. These models would be humorous if it were not for the fact that to employ a smaller number of places in any of the methods in common use would result in loss of accuracy in the final results.

It is important to know under what circumstances a method of determining orbits fails, and the answer to this question is given in full in section 11. In order that it may not be supposed that certain special cases which may arise are peculiar to the method of this paper, it is shown that they occur also in the methods of LAPLACE and of GAUSS; they are, in fact, due to essential peculiarities of the problem.

Another important question is that of the convenience of a method of computing orbits. The chief advantage of that presented here over those in use is that generally nearly all the work can be done with four- or five-place tables without loss of accuracy. The work is so arranged that in the successive approximations all those quantities which depend only on the observed coördinates and the positions of the observer are combined in functions which are computed once for all. In this respect the original method of LAPLACE suffers by comparison with that of GAUSS. The practical features which have been mentioned are desirable, but the differences of opinion representing the merits of other methods indicates that it is inadvisable to make comparisons. BAUSCHINGER concludes* from a very wide experience in computing orbits that the method of GAUSS is much superior to that of LAPLACE and its modifications; and LEUSCHNER is equally sure† that the method of LAPLACE as modified by HARZER and himself is superior to that of GAUSS. It should be remembered in considering such claims that a great deal depends upon the computer's familiarity with all the formulas of a method, and rarely would he have an equally detailed knowledge of just the best arrangement and mode of procedure for two different methods.

2. SOLUTION OF LINEAR EQUATIONS HAVING A SMALL DETERMINANT.

In order not to break the continuity of the argument, a short preliminary discussion of the solution of ordinary linear equations having small determinants will be given. In this discussion the interest will be centered on the question of the degree of accuracy of the final results, and on the possible means of reducing the number of places used in the computations.

Consider first the trivial example

$$(a^2 - b^2)\, x = c^2 - d^2\,,$$

where a, b, c, and d are given, say by observations, accurately to only six places, and where the value of x is required. Suppose further that a and c are nearly equal to b and d respectively. Suppose, for example, that the first two significant figures are the same. If a^2, b^2, c^2, and d^2 are to be computed, seven-place tables must be used in order to secure accuracy in the sixth place. But the differences $a^2 - b^2$ and $c^2 - d^2$ will be known accurately only to four significant figures. Hence x is defined only to four places, though from the logarithm of the quotient $(c^2 - d^2) \div (a^2 - b^2)$ it could be taken out to six places. One might thoughtlessly conclude from the fact that the data are given to six places, that x has been derived to six places, and that the result satisfies the equation to six places, that x is determined to six places. The error of the conclusion is obvious in so simple a problem.

* *Festschrift Heinrich Weber* (1912).

† The Laplacian Orbit Methods, *Proceedings of the Fifth International Congress of Mathematics* (1913), pp. 209-17.

Now suppose the coefficients are factored into

$$a^2 - b^2 = (a-b)(a+b), \qquad c^2 - d^2 = (c-d)(c+d)$$

The differences $a - b$ and $c - d$ will be given only to four places. Consequently no accuracy is gained by using the sums $a + b$ and $c + d$ by which they are multiplied beyond four, or at the most five, places. That is, when the coefficients are factored before the computation is made the value of x can be found by using five-place tables with all the accuracy with which it is defined by the data of the problem; but if the coefficients are not thus factored, seven-place tables must be employed.

There is a general proposition of computation which is important in the present connection. Suppose the product $N_1 N_2 \ldots N_k$ is to be formed and these numbers are given only to $n_1, \ldots, n_k$ significant figures. Suppose n_j is the smallest of $n_1, \ldots, n_k$. Then, in the computation of the product, no accuracy is gained by using more than n_j places (or $n_j + 1$ places so that errors may not accumulate in the last place) in any of the $N_1, \ldots, N_k$, no matter to how many significant figures they may be given.

Suppose x, y, and z are defined by the equations

$$\begin{cases} .34622\,x + .35381\,y + .36518\,z = .24561, \\ .89318\,x + .90274\,y + .91143\,z = .62433, \\ .22431\,x + .23642\,y + .24375\,z = .17145, \end{cases}$$

where the coefficients and right members are furnished by observations, and are subject to errors not exceeding five units in the sixth place. There is no trouble in solving these equations by successive elimination of the unknowns, as is customary in the method of least squares. The result which is obtained with seven-place tables is

$$x = -1.027066, \qquad y = +2.091962, \qquad z = -0.380515.$$

These results satisfy the equations exactly to the last place. Moreover, if either x, y, or z alone is changed by so much as one unit in the fifth decimal place the equations are no longer exactly satisfied. One is tempted to draw the conclusion that x, y, and z are determined accurately to at least five significant figures. But

$$x = -1.023053, \qquad y = +2.082886, \qquad z = -0.375479$$
$$x = -1.031079, \qquad y = +2.101038, \qquad z = -0.385551$$

also exactly satisfy equations whose coefficients and right members differ from these by less than five units in the sixth place. The variation in x is about one per cent. and in z it is two and one-half per cent. That is, the values of x, y, and z are subject to uncertainties which are about one thousand times what might be expected from the first results. The reason for the great uncertainty in the results is that the determinant of the coefficients is small.

In order to investigate the uncertainties in the solutions of linear equations consider the three equations (the discussion is similar for any number).

$$\left.\begin{aligned} a_1 x + b_1 y + c_1 z &= d_1, \\ a_2 x + b_2 y + c_2 z &= d_2, \\ a_3 x + b_3 y + c_3 z &= d_3, \end{aligned}\right\} \tag{1}$$

where the a_i, b_i, c_i, and d_i are supposed to be given by observations to a certain number of places. Suppose the determinant

$$D = \begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix} \tag{2}$$

is distinct from zero. Then, by the theory of linear equations, the solution of (1), regarding the a_i, b_i, c_i, and d_i as exactly known, is unique, finite, and definite. The assumption that the a_i, b_i, c_i, and d_i are exactly known is equivalent to assuming that the digits in the omitted places (supposing the numbers are all written as decimals) are all zero. Let the exact values of the coefficients be $a_i + \alpha_i$, $b_i + \beta_i$, $c_i + \gamma_i$, and $d_i + \delta_i$ ($i = 1, 2, 3$), where the α_i, β_i, γ_i, and δ_i are unknown but do not exceed five units in the first neglected place. Let x_0, y_0, and z_0 represent the solution of equations (1) under the assumption that the coefficients are exact. Let $x_0 + \xi$, $y_0 + \zeta$, $z_0 + \zeta$ be the true values of the unknown quantities. They satisfy the equations

$$\begin{cases} (a_1 + \alpha_1)(x_0 + \xi) + (b_1 + \beta_1)(y_0 + \zeta) + (c_1 + \gamma_1)(z_0 + \zeta) = d_1 + \delta_1, \\ (a_2 + \alpha_2)(x_0 + \xi) + (b_2 + \beta_2)(y_0 + \zeta) + (c_2 + \gamma_2)(z_0 + \zeta) = d_2 + \delta_2, \\ (a_3 + \alpha_3)(x_0 + \xi) + (b_3 + \beta_3)(y_0 + \zeta) + (c_3 + \gamma_3)(z_0 + \zeta) = d_3 + \delta_3. \end{cases}$$

On substracting equations (1) from these equations and solving for ξ, it is found that

$$\xi = \frac{\begin{vmatrix} \delta_1 - \alpha_1 x_0 - \beta_1 y_0 - \gamma_1 z_0, & b_1 + \beta_1, & c_1 + \gamma_1 \\ \delta_2 - \alpha_2 x_0 - \beta_2 y_0 - \gamma_2 z_0, & b_2 + \beta_2, & c_2 + \gamma_2 \\ \delta_3 - \alpha_3 x_0 - \beta_3 y_0 - \gamma_3 z_0, & b_3 + \beta_3, & c_3 + \gamma_3 \end{vmatrix}}{\begin{vmatrix} a_1 + \alpha_1, & b_1 + \beta_1, & c_1 + \gamma_1 \\ a_2 + \alpha_2, & b_2 + \beta_2, & c_2 + \gamma_2 \\ a_3 + \alpha_3, & b_3 + \beta_3, & c_3 + \gamma_3 \end{vmatrix}}. \tag{3}$$

The uncertainty in x due to the unknowns α_i, β_i, γ_i, and δ_i is ξ, which vanishes with these quantities. The problem is to find how large a numerical value ξ may have. Unless D, equation (2), has one or more significant figures distinct from zero the problem is indeterminate, for otherwise the denominator of (3) may be positive, negative, or zero. Suppose D has one or more significant figures not zero; then in determining the limit on the first significant

figure of ξ, which is all that is desired, the α_i, β_i, γ_i of the denominator and of the second and third columns of the numerator may be neglected. Under this hypothesis as to the value of D, at least the first significant figure of ξ is determined by

$$\xi = \frac{\begin{vmatrix} \delta_1 - \alpha_1 x_0 - \beta_1 y_0 - \gamma_1 z_0\,, & b_1\,, & c_1 \\ \delta_2 - \alpha_2 x_0 - \beta_2 y_0 - \gamma_2 z_0\,, & b_2\,, & c_2 \\ \delta_3 - \alpha_3 x_0 - \beta_3 y_0 - \gamma_3 z_0\,, & b_3\,, & c_3 \end{vmatrix}}{D}.$$

The maximum value of ξ is

$$(4) \qquad \xi = \frac{\delta}{D}\left\{1 + |x_0| + |y_0| + |z_0|\right\}\left\{|b_2 c_3 - b_3 c_2| + |b_3 c_1 - b_1 c_3| + |b_1 c_2 - b_2 c_1|\right\},$$

where δ is the common greatest numerical value the α_i, β_i, γ_i, and δ_i can take, and where $|x_0|$, $|b_2 c_3 - b_3 c_2|$, . . . are the numerical values of x_0, $b_2 c_3 - b_3 c_2$, ... There are similar expressions for η and ζ, but ξ, η, and ζ can not in general simultaneously have their greatest values because the signs of all of these quantities are fixed by the condition that ξ, for example, shall have its greatest numerical value.

If in the numerical example which has been considered δ is taken as five units in the sixth place, it is found from (4) that ξ lies between plus and minus five units in the third place. That is, ξ has a range of ten units in the third place. In the two solutions of the problem last given there was a difference in the values of x of eight units in the third place, or only four-fifths of the possible variation. The value of y is uncertain nearly to two units in the second decimal, and z to one unit in the second decimal. It is seen from (4) how a small value of D gives a large uncertainty ξ in the value of x.

An interesting point is illustrated by the numerical example which has been solved. The solution, regarding the digits beyond the fifth place as unknown, is determinate only to two places; yet *the equations are not in general exactly satisfied if x, y, or z is varied in the fifth decimal.* Since any digits in the fifth place in the solution are permissible, the fact that a solution does not exactly satisfy the original equations does not prove that it is not correct so far as it is determinate; and the fact that it does exactly satisfy the equations does not prove that it is correct beyond where it ceases to be determinate. That is, when the determinant of a linear system is small the test of accuracy of the solution by direct substitution loses much of its value. The situation is almost exactly analogous in the theory of orbits where linear systems of equations essentially of the type considered here are involved, though the fact is obscured by the complexity of the ordinary formulas. The determinant is always small when the observations are near together, and sometimes when they are far apart. If the determinant is small the solutions of these linear equations are poorly determined, and the elements are subject to corresponding uncertainties. The fact that the elements may satisfy the observations within the limits of their errors is no proof whatever that they have been determined with a high degree of accuracy. To give under these conditions a model computation with the elements carried out to seconds and tenths of seconds of arc is to violate the ordinary common sense of the problem.

That the solution of a linear system of equations is poorly determined when the determinant of its coefficients is small can be made intuitionally clear by geometrical considerations. For simplicity consider equations (1). Each one taken separately is the equation of a plane; the point of intersection of the three planes is the solution of the equations. The coefficients of x, y, and z are proportional to the direction cosines of the normals to the planes. Consider those normals which pass through the origin. These normals cut the surface of the unit sphere whose center is at the origin in points whose coördinates are proportional to the coefficients of x, y, z; or equal to them when the equations have been multiplied by such factors that in each of them the sum of the squares of the coefficients is unity. Then the determinant D is numerically six times the volume of the tetrahedron whose four vertices are the origin and the three points on the unit sphere. The determinant is small when the volume of the tetrahedron is small, which happens if the three points on the sphere are near together or if they lie nearly on a great circle. In the former case the three planes are nearly parallel, and it is clear from the geometrical situation that large uncertainties in their common point of intersection correspond to slight uncertainties in the positions of the planes. In the second case the line of intersection of any two planes is nearly parallel to the third, and again small displacements of the planes in general produce large changes in their point of intersection.

Since when the determinant of a linear system of equations is small the solution is defined only to a small number of places, the question arises whether it is not possible to make use of this fact in such a way that the computations can be made with shorter tables without impairing the real accuracy of the final results. No such abbreviation can be made if the solution is made by successive elimination of the unknowns, but by determinants it is a simple matter. Suppose the determinants are small because the successive columns are approximately equal, as they are in case of the determinants which arise in the theory of orbits. Then by taking differences of the

columns, or differences after multiplying by simple factors, the determinants can be transformed so that they have one or more small columns. Since in their expansions one factor comes from each term in each column, it is sufficient to use tables having one more place than the number of significant figures in the smallest column.

For example, in the numerical equations which have been considered, the value of x is

$$x = \frac{\begin{vmatrix} .24561\,, & .35381\,, & .36518 \\ .62433\,, & .90274\,, & .91143 \\ .17141\,, & .23642\,, & .24375 \end{vmatrix}}{\begin{vmatrix} .34622\,, & .35381\,, & .36518 \\ .89318\,, & .90274\,, & .91143 \\ .22431\,, & .23642\,, & .24375 \end{vmatrix}}\,.$$

In order to secure the value of x to two places from these determinants as they stand, it is necessary to use six-place tables because of the cancellation, in both numerator and denominator of approximately equal terms having opposite signs. This corresponds exactly to the process of the successive elimination of the unknowns. But if in the numerator determinant two-thirds of the third column is substracted from the first column, and if the second column is substracted from the third column and if the resulting third column is subtracted from the resulting first column; and if in the denominator determinant the second column is subtracted from both the first and third columns, and if the resulting third column is added to the resulting first column, the values of the determinants are not changed, and the expression for x becomes

$$x = \frac{\begin{vmatrix} -.00921\,, & .35381\,, & .01137 \\ +.00802\,, & .90274\,, & .00869 \\ +.00158\,, & .23642\,, & .00733 \end{vmatrix}}{\begin{vmatrix} +.00378\,, & .35381\,, & .01137 \\ -.00087\,, & .90274\,, & .00869 \\ -.00478\,, & .23642\,, & .00733 \end{vmatrix}}\,.$$

Since the first columns contain only three significant figures it is sufficient in the computation to use four-place tables in which all the numbers are given on two pages. The result to three places is $x = -1.03$, and this is as far as the solution is definite and is as accurate a value as can be obtained from the data with any number of places. The determinant expressions for y and z can be treated similarly.

Suppose the solution of the linear equations is computed from the transformed determinants by abbreviated tables depending on the degree of determination of the result, and let it be x_0, y_0, z_0. This solution will not in general satisfy the determining equations to more places than it is given. Suppose the left members of (1) give $d_1^{(0)}$, $d_2^{(0)}$, and $d_3^{(0)}$ when the solution x_0, y_0, z_0 is substituted in them. Let $d_i - d_i^{(0)} = \delta_i^{(0)}$, which in general will be small compared to d_i. Let $x = x_0 + \xi$, $y = y_0 + \eta$, $z = z_0 + \zeta$ be a solution which satisfies equations (1) to the last place. On substracting the corresponding equations in x_0, y_0, and z_0, with $d_i^{(0)}$ in the right members, it is found that ξ, η, and ζ are defined by

$$\left.\begin{aligned} a_1\,\xi + b_1\,\eta + c_1\,\zeta &= \delta_1^{(0)}\,, \\ a_2\,\xi + b_2\,\eta + c_2\,\zeta &= \delta_2^{(0)}\,, \\ a_3\,\xi + b_3\,\eta + c_3\,\zeta &= \delta_3^{(0)}\,. \end{aligned}\right\} \qquad (5)$$

These equations can also be computed by the use of the short tables. The denominator determinant is already known from the preceding computation, and if the $\delta_i^{(0)}$ are very small a very few places in the tables are sufficient for computing the numerator determinants. If $x_0 + \xi$, $y_0 + \eta$, and $z_0 + \zeta$ do not exactly satisfy equations (1) the process may be repeated. If the determinant D is very small it may well be necessary to do so. Eventually results will be obtained which will exactly satisfy the original equations, but they are not in general exact beyond the point where they cease to be determinate, that is beyond x_0, y_0, and z_0.

One might ask why these supplementary solutions should be made, even though they secure results which exactly satisfy the equations, if they belong to a part of the solution which is essentially indeterminate. Obviously there is no sensible reason for going to the trouble of making them. But the corresponding thing is done in the determination of orbits, and it is for the purpose of making the matter clear that it is considered in this simple problem where it is not obscured by complicated formulas and a multitude of supplementary considerations. The fact is that all methods of determining orbits from three observations which are near together involve a partial indetermination which is exactly comparable to the one under consideration here. The indetermination is fundamentally in the problem and can not be avoided. This means that the elements are determined in this way only approximately. But when the elements are given there is no indetermination in computing the apparent position of the body. Consequently, if the elements are computed only to the extent that they are determinate, they will in general not give results in exact agreement with the observations. The reason is that not all of the six-fold infinity of sets of elements within the limits of their determination are compatible with the observations; but there are infinitely many sets of elements which give theoretical positions agreeing with the observational. To make the essence of the matter clear consider the linear system (1). When the determinant of its coefficients is small x, y, and z are determined only to a small number of places — in the numerical example to three. But x, y, and z can not be given *arbitrary values* beyond the points at which they

cease to be determinate, for the numerical values of the left members of (1) are sensitive to individual changes in the unknowns. Similarly, while the elements of an orbit are poorly determined by observations which are close together, the theoretical positions are sensitive to changes in the individual elements. Though the elements may not be changed individually beyond the point at which they cease to be determinate, if the observations are to be satisfied, yet they can be changed collectively in infinitely many ways without destroying the harmony between theory and observation.

3. The Geometrical Conditions.

Suppose the observations are made at t_1, t_2, and t_3. Let the corresponding polar coördinates with the observer as the origin be respectively $(\rho_1, \alpha_1, \delta_1)$, $(\rho_2, \alpha_2, \delta_2)$ and $(\rho_3, \alpha_3, \delta_3)$, where the ρ_i are the unknown distances. In general the observations will be expressed directly in α and δ, the right ascension and declination. Let the direction cosines of the observed body be represented by λ, μ, and ν, with subscripts corresponding to those on the t. Therefore

$$(6) \qquad \left\{\begin{aligned} \lambda_i &= \cos\delta_i \cos\alpha_i\,, \quad (i = 1, 2, 3), \\ \mu_i &= \cos\delta_i \sin\alpha_i\,, \\ \nu_i &= \sin\delta_i\,. \end{aligned}\right.$$

It is supposed that the α_i, and δ_i have been corrected for precession, etc.

Let the rectangular geocentric coördinates of the Sun referred to the equator system be $X_i^{(0)}$, $Y_i^{(0)}$, and $Z_i^{(0)}$. They are of course given in the *American Ephemeris*. Let ΔX_i, ΔY_i, and ΔZ_i represent the geocentric coördinates of the observer referred to the same axes. Let θ_i represent the observer's sidereal time at t_i, l his geocentric latitude, and r_E the radius of the Earth. It will generally be sufficiently accurate to use the mean radius of the Earth, though the radius to the observer is the exact quantity in question. It follows that r_E, θ_i and l are the geocentric polar coördinates of the observer referred to the same axes. Then

$$(7) \qquad \left\{\begin{aligned} \Delta X_i &= r_E \cos l \cos\theta_i\,, \quad (i = 1, 2, 3), \\ \Delta Y_i &= r_E \cos l \sin\theta_i\,, \\ \Delta Z_i &= r_E \sin l\,. \end{aligned}\right.$$

If the observations are made at different places corresponding different values of r_E and l must be used.

Let X_i, Y_i, and Z_i represent the heliocentric coördinates of the observer. They are given by

$$(8) \qquad \left\{\begin{aligned} X_i &= -X_i^{(0)} + \Delta X_i\,, \quad (i = 1, 2, 3), \\ Y_i &= -Y_i^{(0)} + \Delta Y_i\,, \\ Z_i &= -Z_i^{(0)} + \Delta Z_i\,. \end{aligned}\right.$$

Finally, let x_i, y_i, and z_i represent the heliocentric coördinates of the observed body at the epoch t_i. They are respectively equal to the sums of the corresponding heliocentric coördinates of the observer and the coördinates of the observed body with respect to the observer as an origin. Hence the geometrical conditions to which the coördinates of the observed body are subject are

$$\left.\begin{aligned} -\lambda_i\,\rho + x_i &= X_i\,, \quad (i = 1, 2, 3), \\ -\mu_i\,\rho + y_i &= Y_i\,, \\ -\nu_i\,\rho + z_i &= Z_i\,. \end{aligned}\right\} \qquad (9)$$

These equations are subject to no errors of parallax because the geocentric coördinates of the observer have been introduced. The only correction that remains to be made to them is for the time it takes light to go from the observed body to the observer. Matters will be arranged so that it will be necessary to make almost no alterations in the computed quantities in applying this correction.

4. The Dynamical Conditions.

It will be assumed that the observed body revolves about the Sun, though the formulas all hold whatever may be the center of attraction. Let

$$k\sqrt{1+m}\,(t - t_0) = \tau\,. \qquad (10)$$

where $1 + m$ is the sum of masses of the Sun and observed body, and t_0 is the origin of time. Then the differential equations of motion are

$$\left.\begin{aligned} \frac{d^2x}{d\tau^2} &= -\frac{x}{r^3} = -u\,x\,, \quad (r^2 = x^2 + y^2 + z^2 = u^{-\frac{2}{3}})\,, \\ \frac{d^2y}{d\tau^2} &= -\frac{y}{r^3} = -u\,y\,, \\ \frac{d^2z}{d\tau^2} &= -\frac{z}{r^3} = -u\,z\,. \end{aligned}\right\} \qquad (11)$$

If any other body than the Sun has sensible effects the corresponding terms can be added to the differential equations; it causes no difficulty except the greater complexity of the formulas. In general, a comet or planetoid is not subject to great enough disturbing forces to make it advisable to take them into account; but under the special conditions in which the observational data are very accurate and the perturbing forces are very large it might be advantageous to take them into account. The reason is that while their results would be small in the short interval covered by the observations, the elements are very sensitive to slight differences in observed positions when the observations are near together.

Inasmuch as the observed body will not be at infinity or in collision with the Sun, it follows from the general

theory of analytic differential equations that the solution of (11) is developable as a power series in τ of the form

$$(12)\quad \begin{cases} x = x_0 + x_0'\,\tau + \frac{1}{2}x_0''\,\tau^2 + \frac{1}{6}x_0'''\,\tau^3 + \ldots\,, \\ y = y_0 + y_0'\,\tau + \frac{1}{2}y_0''\,\tau^2 + \frac{1}{6}y_0'''\,\tau^3 + \ldots\,, \\ z = z_0 + z_0'\,\tau + \frac{1}{2}z_0''\,\tau^2 + \frac{1}{6}z_0'''\,\tau^3 + \ldots\,. \end{cases}$$

The domain of convergence of these series depends only upon $t_0 - T$, which is the time from perihelion passage to the origin of time, the major axis, and the eccentricity of the orbit. These elements of course are not known in advance, and hence the range of validity of the series is not known in advance. But a complete discussion of the singularities of the functions, which define the true circles of convergence for any origin, was given by the writer in the *Astronomical Journal*, Vol. XXIII, Nos. 537-538 (1903), where a solid foundation was laid for all work in this domain. It was there shown that the circles of convergence of (12) are smaller, the smaller the major axis (parameter if $e = 1$), the greater the eccentricity, and the nearer the body is to its perihelion at t_0. Tables III and IV of that paper give by inspection the general order of the results which arise in practice. If the major axis is 2.65 astronomical units and the eccentricity does not exceed 0.4, numbers suggested by the planetoids, the series converge when the body is in any part of its orbit if $t - t_0$ does not exceed one hundred and sixty-three days. If the orbit is a parabola whose perihelion distance is unity, the series converge if $t - t_0$ does not exceed fifty-four days. These limits vary directly as the three-halves power of the major axis and perihelion distance respectively. In practice the series must not only converge but they must converge rapidly. It is evident from the results which have been given that they will satisfy this condition for the short intervals which are ordinarily used in the determination of orbits.

It is found by forming the successive derivatives of (11) that equations (12) become

$$(13)\quad \begin{cases} x = f\,x_0 + g\,x'_0\,, \\ y = f\,y_0 + g\,y'_0\,, \\ z = f\,z_0 + g\,z_0'\,, \\ f = 1 - \frac{1}{2}u\tau^2 - \frac{1}{6}u'\tau^3 - \frac{1}{24}(u'' - u^2)\tau^4 - \frac{1}{120}(u''' - 4uu')\tau^5 + \ldots\,, \\ g = \tau - \frac{1}{6}u\tau^3 - \frac{1}{12}u'\tau^4 - \frac{1}{120}(3u'' - u^2)\tau^5 - \frac{1}{360}(2u''' - 3uu')\tau^6 + \ldots\,, \end{cases}$$

where now x_0, y_0, z_0 enter in u instead of x, y, z. LAGRANGE has expressed the derivatives of u in terms of u and two auxiliaries* p and q defined by

$$(14)\quad \begin{cases} r^2p = \frac{1}{2}\dfrac{dr^2}{d\tau} = x_0\,x_0' + y_0\,y_0' + z\,z'_0\,, \\ r^2q = \frac{1}{2}\dfrac{d^2r^2}{d\tau^2} = x_0'^2 + y_0'^2 + z_0'^2 - r_0^2u\,. \end{cases}$$

Since $2r^2p$ and $2r^2q$ are the first and second derivatives of r^2 they vanish when the observed body is at an apse of its orbit, and they are permanently zero in the case of a circular orbit. In the asteroid orbits they are in general small because they carry the eccentricity as a factor.

The expressions for f and g become as a consequence of equations (14)

$$(15)\quad \begin{cases} f = 1 - \frac{1}{2}u\tau^2 + \frac{1}{2}up\tau^3 + \frac{1}{24}(3uq - 15up^2 + u^2)\tau^4 + \frac{1}{8}(7up^3 - 3upq - u^2p)\tau^5 + \ldots\,, \\ g = \tau - \frac{1}{6}u\tau^3 + \frac{1}{4}up\tau^4 + \frac{1}{120}(9uq - 45up^2 + u^2)\tau^5 + \frac{1}{24}(14up^3 - 6upq - u^2p)\tau^6 + \ldots\,. \end{cases}$$

Consequently, when $x_0, y_0, z_0', y_0', z_0'$ are known, the series f and g are known for any value of τ for which they converge. Their values at t_1, t_2, and t_3 can be indicated by the corresponding subscripts.

On using equations (13) equations (9) become

$$(16)\quad \begin{cases} -\lambda_1\,\rho_1 + f_1\,x_0 + g_1\,x_0' = X_1\,, \\ -\lambda_2\,\rho_2 + f_2\,x_0 + g_2\,x_0' = X_2\,, \\ -\lambda_3\,\rho_3 + f_3\,x_0 + g_3\,x_0' = X_3\,; \\ -\mu_1\,\rho_1 + f_1\,y_0 + g_1\,y_0' = Y_1\,, \\ -\mu_2\,\rho_2 + f_2\,y_0 + g_2\,y_0' = Y_2\,, \\ -\mu_3\,\rho_3 + f_3\,y_0 + g_3\,y_0' = Y_3\,; \\ -\nu_1\,\rho_1 + f_1\,z_0 + g_1\,z_0' = Z_1\,, \\ -\nu_2\,\rho_2 + f_2\,z_0 + g_2\,z_0' = Z_2\,, \\ -\nu_3\,\rho_3 + f_3\,z_0 + g_3\,z_0' = Z_3\,. \end{cases}$$

In these equations x_0, y_0, z_0, x_0', y_0', and z_0' are entirely unknown. When they have been found the elements can be determined without ambiguity or numerical indetermination. The only exception to this statement is that the perihelion is poorly determined if the eccentricity is small, and the nodes are subject to large uncertainties if the inclination is small. The distances ρ_1, ρ_2, and ρ_3 are also unknown. It would not be necessary to determine them if it were not for correcting for the time required for light to go from the observed body to the observer. Only the first terms of the series f_1, f_2, f_3, g_1, g_2, and g_3 are known, but the coefficients of the terms of higher degree involve only x_0, y_0, z_0, x_0', y_0', and z_0'. Thus in the nine equations (16) there are exactly nine unknown quantities. The problem is to solve them for these unknowns.

Oeuvres de LAGRANGE, Vol. IV, pp. 500, 514; CHARLIER, *Meddelande från Lunds Astro. Obs.*, No. 46.

To recapitulate, equations (9) are the complete expressions of the exact geometrical relations which are involved, and every method of determining orbits depends either upon them or upon some of their special consequences. Equations (13) are the complete expressions of the dynamical relations, developed in series. All methods involve corresponding series. And in addition to them, the original method of LAPLACE and its modifications by HARZER and LEUSCHNER contain corresponding developments for the coördinates of the observer. This is a defect of these methods because logically the *motion* of the observer has nothing to do with the orbit of the observed body — only the positions of the observer are really involved. But in the Laplacian methods it is necessary to develop series for the motion of the observer. This motion depends upon the revolution and rotation of the Earth, the fact that possibly the observations have been made at different observatories, as POINCARÉ has remarked*; and the fact that the center of gravity of the Earth and Moon describes the Earth's elliptic orbit, as BRUNS has insisted.† The alternative is to make complete, and sometimes repeated, corrections for parallax as LEUSCHNER has done in his computations.‡ Whichever of these methods is employed it is necessary to obtain not only the position of the observer at the epoch, but also at least the components of velocity of the Earth.

5. THE DETERMINANT OF THE COEFFICIENTS OF (16).

Since the f_i and the g_i are approximately known when the time-intervals are short, it might be supposed that if all but their first terms were neglected the solution of (16) would give approximate values of the unknowns ρ_1, ρ_2, ρ_3, x_0, x_0', y_0, y_0', z_0, and z_0', which enter linearly. The degree of approximation in the result depends to a large extent upon the determinant of the coefficients. And whether the solution is approximate or not there can be no objection to solving equations for the unknowns so far as they appear explicitly (linearly), for this process is equivalent to forming linear combinations of the original equations. The only precaution necessary is not to divide through by the determinant of the coefficients in case it is zero. Consequently, the first problem is the discussion of the determinant of the coefficients of (16).

Suppose the unknowns in (16) are arranged from left to right in the order ρ_1, ρ_2, ρ_3, x_0, x_0', y_0, y_0', z_0, and z_0'. Suppose also that t_2 is taken as the origin of time, or $t_0 = t_2$. Then $\tau_2 = 0$, $g_2 = 0$, and $f_2 = 1$, and the determinant of the coefficients of the unknown quantities is

$$\Delta = -\begin{vmatrix} \lambda_1, & 0, & 0, & f_1, & g_1, & 0, & 0, & 0, & 0 \\ 0, & \lambda_2, & 0, & 1, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & \lambda_3, & f_3, & g_3, & 0, & 0, & 0, & 0 \\ \mu_1, & 0, & 0, & 0, & 0, & f_1, & g_1, & 0, & 0 \\ 0, & \mu_2, & 0, & 0, & 0, & 1, & 0, & 0, & 0 \\ 0, & 0, & \mu_3, & 0, & 0, & f_3, & g_3, & 0, & 0 \\ \nu_1, & 0, & 0, & 0, & 0, & 0, & 0, & f_1, & g_1 \\ 0, & \nu_2, & 0, & 0, & 0, & 0, & 0, & 1, & 0 \\ 0, & 0, & \nu_3, & 0, & 0, & 0, & 0, & f_3, & g_3 \end{vmatrix}. \tag{17}$$

Some general properties of Δ follow immediately from (17). It is homogeneous of the third degree in the λ_i, μ_i, and ν_i, and of the third degree in g_1 and g_3 because each term of the expansion has one factor from each column. It is a symmetrical function of λ_i, μ_i, and ν_i because if these letters are cyclically permuted the original form of the determinant can be restored by an even number of permutations of rows and of columns. There is no term which involves $\lambda_1\lambda_2\lambda_3$ as a factor because the co-factor of the third order minor which gives this product has two columns of zeros. There is no term which involves $\lambda_1\lambda_2$ as a factor because the co-factor of the second order minor which gives this product has two columns whose elements are all zero except the first. It follows from these facts and the symmetry of Δ that every term of its expansion has a factor $\lambda_i\mu_j\nu_k$, where no two of the i, j, k are the same.

When g_1 is put equal to zero, Δ becomes

$$-\begin{vmatrix} \lambda_1, & 0, & 0, & f_1, & 0, & 0, & 0, & 0, & 0 \\ 0, & \lambda_2, & 0, & 1, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & \lambda_3, & f_3, & g_3, & 0, & 0, & 0, & 0 \\ \mu_1, & 0, & 0, & 0, & 0, & f_1, & 0, & 0, & 0 \\ 0, & \mu_2, & 0, & 0, & 0, & 1, & 0, & 0, & 0 \\ 0, & 0, & \mu_3, & 0, & 0, & f_3, & g_3, & 0, & 0 \\ \nu_1, & 0, & 0, & 0, & 0, & 0, & 0, & f_1, & 0 \\ 0, & \nu_2, & 0, & 0, & 0, & 0, & 0, & 1, & 0 \\ 0, & 0, & \nu_3, & 0, & 0, & 0, & 0, & f_3, & g_3 \end{vmatrix} = g_3{}^3 \begin{vmatrix} \lambda_1, & 0, & 0, & f_1, & 0, & 0 \\ 0, & \lambda_2, & 0, & 1, & 0, & 0 \\ \mu_1, & 0, & 0, & 0, & f_1, & 0 \\ 0, & \mu_2, & 0, & 0, & 1, & 0 \\ \nu_1, & 0, & 0, & 0, & 0, & f_1 \\ 0, & \nu_2, & 0, & 0, & 0, & 1 \end{vmatrix}.$$

This determinant is zero because the elements of the third column are all zero. Therefore g_1 is a factor of Δ. Similarly, g_3 and also $f_1 g_3 - f_3 g_1$ are factors of Δ. Therefore

$$\Delta = -g_1 g_3 (f_1 g_3 - f_3 g_1)\Delta_2, \tag{18}$$

where Δ_2 involves only the λ_i, μ_i, and ν_i.

Let $F = 1 + f_1 + f_3$, $G = g_1 + g_3$. Suppose the second and third rows of (17) are added to the first, the fourth and sixth to the fifth, and the seventh and eighth to the ninth. Then, if the resulting fifth and ninth rows are brought to the second and third row places respectively, the determinant takes the form

$$\Delta = +\begin{vmatrix} \lambda_1, & \lambda_2, & \lambda_3, & F, & G, & 0, & 0, & 0, & 0 \\ \mu_1, & \mu_2, & \mu_3, & 0, & 0, & F, & G, & 0, & 0 \\ \nu_1, & \nu_2, & \nu_3, & 0, & 0, & 0, & 0, & F, & G \\ 0, & \lambda_2, & 0, & 1, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & \lambda_3, & f_3, & g_3, & 0, & 0, & 0, & 0 \\ \mu_1, & 0, & 0, & 0, & 0, & f_1, & g_1, & 0, & 0 \\ 0, & 0, & \mu_3, & 0, & 0, & f_3, & g_3, & 0, & 0 \\ \nu_1, & 0, & 0, & 0, & 0, & 0, & 0, & f_1, & g_1 \\ 0, & \nu_2, & 0, & 0, & 0, & 0, & 0, & 1, & 0 \end{vmatrix}. \tag{19}$$

* *Bulletin Astronomique*, Vol. XXIII (1906), pp. 161–187.

† BAUSCHINGER, *Bahnbestimmung*, p. 347.

‡ *Publications of the Lick Observatory*, Vol. VII, Part I.

The leading third-order minor of (19) involves the λ_i, μ_i, and ν_i in precisely the form that it has been shown they occur in the expansion of Δ. Moreover, the co-factor of this minor is the coefficient of Δ_2 in (18). It will be shown that the determinant equals precisely the product of this third-order minor and its co-factor, by proving that the coefficient of every term which involves a λ_i, μ_i, or ν_i not occurring in the first three rows is zero. For this purpose consider the term $\lambda_2\mu_1\nu_3 K$, where K is a function of the f_i and g_i and where at least one of λ_2, μ_1, ν_3 comes from the last six rows. There are three of these terms, viz., λ_2 from the first row, μ_1 from the sixth row, and ν_3 from the third row; λ_2 from the fourth row, μ_1 from the second row, and ν_3 from the third row; and λ_2 from the fourth row, μ_1 from the sixth row, and ν_3 from the third row. The coefficient K is the sum of the three determinants which remain when the columns and rows from which λ_2, μ_1, and ν_3 have been taken are suppressed, the signs being determined by the general rule of signs of co-factors. The resulting expression for K is

$$\begin{vmatrix} 0, & 0, & F, & G, & 0, & 0 \\ 1, & 0, & 0, & 0, & 0, & 0 \\ f_3, & g_3, & 0, & 0, & 0, & 0 \\ 0, & 0, & f_3, & g_3, & 0, & 0 \\ 0, & 0, & 0, & 0, & f_1, & g_1 \\ 0, & 0, & 0, & 0, & 1, & 0 \end{vmatrix} + \begin{vmatrix} F, & G, & 0, & 0, & 0, & 0 \\ f_3, & g_3, & 0, & 0, & 0, & 0 \\ 0, & 0, & f_1, & g_1, & 0, & 0 \\ 0, & 0, & f_3, & g_3, & 0, & 0 \\ 0, & 0, & 0, & 0, & f_1, & g_1 \\ 0, & 0, & 0, & 0, & 1, & 0 \end{vmatrix} + \begin{vmatrix} F, & G, & 0, & 0, & 0, & 0 \\ 0, & 0, & F, & G, & 0, & 0 \\ f_3, & g_3, & 0, & 0, & 0, & 0 \\ 0, & 0, & f_3, & g_3, & 0, & 0 \\ 0, & 0, & 0, & 0, & f_1, & g_1 \\ 0, & 0, & 0, & 0, & 1, & 0 \end{vmatrix}$$

The first and third of these determinants can be added, after the first two rows of the first are interchanged, by taking the sum of the elements of the first rows. The result is the negative of the second determinant because it becomes identical with it by an even number of permutations of rows and of columns, and by expressing F and G in terms of the f_i and the g_i. Consequently K is zero. In a similar manner it can be shown that the coefficient of every term which involves at least one of λ_i, μ_i, ν_i from the last six rows is zero. Therefore the determinant Δ is

$$(20)\quad \Delta = -g_1g_3(f_1g_3 - f_3g_1)\begin{vmatrix} \lambda_1, & \lambda_2, & \lambda_3 \\ \mu_1, & \mu_2, & \mu_3 \\ \nu_1, & \nu_2, & \nu_3 \end{vmatrix} = -g_1g_3(1,3)\Delta_2 .$$

It follows from (15) that

(21)

$$\begin{cases} g_1 = \tau_1 - \frac{1}{6}u\tau_1^3 + \frac{1}{4}up\tau_1^4 + \frac{1}{120}(9uq - 45up^2 \\ \qquad + u^2)\tau_1^5 + \frac{1}{24}(14up^3 - 6upq - u^2p)\tau_1^6 + \ldots, \\ g_3 = \tau_3 - \frac{1}{6}u\tau_3^3 + \frac{1}{4}up\tau_3^4 + \frac{1}{120}(9uq - 45up^2 \\ \qquad + u^2)\tau_3^5 + \frac{1}{24}(14up^3 - 6upq - u^2p)\tau_3^6 + \ldots. \\ (1,3) = (\tau_3 - \tau_1)\{1 - \frac{1}{6}u(\tau_3 - \tau_1)^2 + \frac{1}{4}up(\tau_3 - \tau_1)^2 \\ \qquad (\tau_3 + \tau_1) + \ldots\}. \end{cases}$$

Therefore since τ_1 and τ_3 are distinct from zero, but small and opposite in sign, Δ is not zero unless Δ_2 is zero. Since $\rho_1\lambda_1 = \xi_1$, $\rho_1\mu_1 = \eta_1$, ..., it follows that

$$\rho_1\rho_2\rho_3\Delta_2 = \begin{vmatrix} \xi_1, & \xi_2, & \xi_3 \\ \eta_1, & \eta_2, & \eta_3 \\ \zeta_1, & \zeta_2, & \zeta_3 \end{vmatrix}, \qquad (22)$$

where the ξ_i, η_i, ζ_i are the rectangular coördinates of the observed body at t_i with respect to the observer as an origin. This determinant is numerically six times the volume of the tetrahedron whose vertices are the observer and the three positions of the body relative to the observer. Therefore Δ_2 is distinct from zero unless the three observed positions of the body lie on the arc of a great circle.

6. The Determinant Δ_2 in the Methods of Gauss and Laplace.

The fact that the case is exceptional and the treatment must be modified if Δ_2 is zero at once raises the question whether or not the difficulty is not artificial, and whether it arises in other methods of determining orbits. The same determinant appears in the method of Gauss, though usually in a more complicated form. It has been clearly and fully treated by Bauschinger, in his *Bahnbestimmung*, p. 258, where it is represented by K. But in the method of Laplace, it does not explicitly appear, especially where approximate expressions for the derivatives are used, though it is involved as will be shown.

In the method of Laplace and its modifications the determinant

$$D = \begin{vmatrix} \lambda, & \lambda', & \lambda'' \\ \mu, & \mu', & \mu'' \\ \nu, & \nu', & \nu'' \end{vmatrix}$$

arises, where the single and double accents indicate first and second derivatives respectively. The values of the direction cosines λ, μ, ν and their first two derivatives are to be determined from three sets of values of λ, μ, ν furnished by the observations. The value of λ for any τ not too remote from the epoch is given by

$$\lambda = \frac{(\tau - \tau_2)(\tau - \tau_3)}{(\tau_1 - \tau_2)(\tau_1 - \tau_3)}\lambda_1 + \frac{(\tau - \tau_1)(\tau - \tau_3)}{(\tau_2 - \tau_1)(\tau_2 - \tau_3)}\lambda_2 + \frac{(\tau - \tau_1)(\tau - \tau_2)}{(\tau_3 - \tau_1)(\tau_3 - \tau_2)}\lambda_3 .$$

There are corresponding expressions for μ and ν having precisely the same coefficients. The values of λ, λ', and λ'' for $\tau = 0$, for which D is used, are

$$\left\{\begin{aligned}
\lambda &= P_1\,\lambda_1 + P_2\,\lambda_2 + P_3\,\lambda_3\,,\\
\lambda' &= P_1'\,\lambda_1 + P_2'\,\lambda_2 + P_3'\,\lambda_3\,,\\
\lambda'' &= P_1''\,\lambda_1 + P_2''\,\lambda_2 + P_3''\,\lambda_3\,,\\
P_1 &= \frac{\tau_2\,\tau_3}{(\tau_1-\tau_2)\,(\tau_1-\tau_3)}\,, \quad P_2 = \frac{\tau_1\,\tau_3}{(\tau_2-\tau_1)\,(\tau_2-\tau_3)}\,,\\
P_1' &= \frac{-\,(\tau_2+\tau_3)}{(\tau_1-\tau_2)\,(\tau_1-\tau_3)}\,, \quad P_2' = \frac{-\,(\tau_1+\tau_3)}{(\tau_2-\tau_1)\,(\tau_2-\tau_3)}\,,\\
P_1'' &= \frac{2}{(\tau_1-\tau_2)\,(\tau_1-\tau_3)}\,, \quad P_2'' = \frac{2}{(\tau_2-\tau_1)\,(\tau_2-\tau_3)}\,,\\
P_3 &= \frac{\tau_1\,\tau_2}{(\tau_3-\tau_1)\,(\tau_3-\tau_2)}\,,\\
P_3' &= \frac{-\,(\tau_1+\tau_2)}{(\tau_3-\tau_1)\,(\tau_3-\tau_2)}\,,\\
P_3'' &= \frac{2}{(\tau_3-\tau_1)\,(\tau_3-\tau_2)}\,.
\end{aligned}\right.$$

Then the determinant D becomes

$$D = \begin{vmatrix} P\lambda\,, & P\lambda'\,, & P\lambda'' \\ P\mu\,, & P\mu'\,, & P\mu'' \\ P\nu\,, & P\nu'\,, & P\nu'' \end{vmatrix}\,,$$

where

$$\left\{\begin{aligned}
P\ \lambda &= P_1\ \lambda_1 + P_2\ \lambda_2 + P_3\ \lambda_3\,,\\
P'\ \lambda &= P_1'\,\lambda_1 + P_2'\,\lambda_2 + P_3'\,\lambda_3\\
P''\lambda &= P_1''\lambda_1 + P_2''\lambda_2 + P_3''\lambda_3
\end{aligned}\right.$$

The determinant D factors into

$$D = \begin{vmatrix} P_1\,, & P_1'\,, & P_1'' \\ P_2\,, & P_2'\,, & P_2'' \\ P_3\,, & P_3'\,, & P_3'' \end{vmatrix} \times \begin{vmatrix} \lambda_1\,, & \lambda_2\,, & \lambda_3 \\ \mu_1\,, & \mu_2\,, & \mu_3 \\ \nu_1\,, & \nu_2\,, & \nu_3 \end{vmatrix}\,.$$

The second of these two factors is Δ_2, and therefore the method of LAPLACE is subject to the same difficulty, when Δ_2 vanishes, that the method developed here and that of GAUSS encounter. The first of the two factors of D reduces to

$$\frac{+\,2}{(\tau_2-\tau_1)\,(\tau_3-\tau_1)\,(\tau_3-\tau_2)}\,,$$

which is finite and distinct from zero.

7. PROOF THAT Δ_2 VANISHES PERMANENTLY ONLY IF THE OBSERVED BODY MOVES ALONG THE ECLIPTIC.

It follows from the geometrical interpretation of Δ_2 that it is always zero if the observed body moves along the ecliptic. It may be zero if this condition is not satisfied, for it is possible for three observed positions to lie on the arc of a great circle; but it will be shown that it is not possible for all its positions to lie on the arc of a great circle with respect to the Earth. In this discussion the position of the observed body will be taken with respect to the center of the Earth rather than with respect to some point on the surface of the Earth.

The geocentric coördinates of the observed body are denoted by ξ, η, ζ, its heliocentric coördinates by x, y, z, and the heliocentric coördinates of the Earth by X, Y, Z. Therefore $\xi = x - X\,, \eta = y - Y\,, \zeta = z - Z$. The condition that the apparent motion of the observed body shall be in the arc of a great circle, that is, that its orbit relative to the Earth shall be in a plane passing through the Earth is

$$a\xi + b\eta + c\zeta = a(x-X) + b(y-Y) + c(z-Z) = 0\,, \quad (23)$$

where a, b, and c are constants. Since this equation holds for all values of the time its second derivative also is zero. Hence

$$a(x'' - X'') + b(y'' - Y'') + c(z'' - Z'') = 0\,. \quad (24)$$

But since both the observed body and the Earth move around the Sun in accordance with the law of gravitation, it follows that

$$x'' = -\frac{k^2\,x}{r^3}\,, \qquad X'' = -\frac{k^2\,X}{R^3}\,,\ \ldots\,,$$

where for simplicity in the formulas it is supposed that the masses of the Earth and observed body are negligible in comparison to that of the Sun. Then equation (24) becomes

$$a\left[\frac{x}{r^3} - \frac{X}{R^3}\right] + b\left[\frac{y}{r^3} - \frac{Y}{R^3}\right] + c\left[\frac{z}{r^3} - \frac{Z}{R^3}\right] = 0\,. \quad (25)$$

It follows from (23) and (25) that

$$a \div b \div c = \begin{vmatrix} y - Y\,, & z - Z \\ \dfrac{y}{r^3} - \dfrac{Y}{R^3}\,, & \dfrac{z}{r^3} - \dfrac{Z}{R^3} \end{vmatrix}$$

$$\div \begin{vmatrix} z - Z\,, & x - X \\ \dfrac{z}{r^3} - \dfrac{Z}{R^3}\,, & \dfrac{x}{r^3} - \dfrac{X}{R^3} \end{vmatrix} \div \begin{vmatrix} x - X\,, & y - Y \\ \dfrac{x}{r^3} - \dfrac{X}{R^3}\,, & \dfrac{y}{r^3} - \dfrac{Y}{R^3} \end{vmatrix}\,.$$

But

$$\begin{vmatrix} y - Y\,, & z - Z \\ \dfrac{y}{r^3} - \dfrac{Y}{R^3}\,, & \dfrac{z}{r^3} - \dfrac{Z}{R^3} \end{vmatrix} = \begin{vmatrix} y\,, & z \\ Y\,, & Z \end{vmatrix} \left(\frac{1}{r^3} - \frac{1}{R^3}\right)\,,$$

and similar reductions for the other determinants. Therefore the proportion becomes

$$a \div b \div c = \begin{vmatrix} y\,, & z \\ Y\,, & Z \end{vmatrix} : \begin{vmatrix} z\,, & x \\ Z\,, & X \end{vmatrix} : \begin{vmatrix} x\,, & y \\ X\,, & Y \end{vmatrix}\,. \quad (26)$$

It follows from this proportion that x, y, z and X, Y, Z, satisfy the equations

$$\begin{cases} a\,x + b\,y + c\,z = 0\,, \\ a\,X + b\,Y + c\,Z = 0\,. \end{cases}$$

That is, equations (23) imply that the observed body and the Earth move in the same plane with respect to the Sun. Hence the only case in which the observed body moves permanently in the arc of a great circle is that in which the plane of its orbit is the same as that of the Earth. If Δ_2 should be zero for three observations of a body not moving in the plane of the ecliptic, there would be other epochs at which it would not be zero. The problem of determining the orbit of a body moving in the plane of the ecliptic requires separate treatment.

8. Solution of Equations (16) for ρ_1, ρ_2, and ρ_3.

It is necessary to solve equations (16) for x_0, y_0, z_0, x_0', y_0', and z_0' in order to determine the elements, and for ρ_1, ρ_2, and ρ_3 in order to correct for the time aberration. It is simplest to solve first for ρ_1, ρ_2, ρ_3. Since the origin of time is taken so that $t_0 = t_2$ it follows that $x_0 = x_2$, $y_0 = y_2$, . . . , $z_0' = z_2'$, $f_2 = 1$, $g_2 = 0$. Therefore after ρ_1, ρ_2, and ρ_3 have been computed, x_2, y_2, and z_2 can be uniquely determined from the second, fourth, and eighth equations respectively; and then the derivatives can be found from the remaining equations with the greatest ease because only one occurs in each equation.

The solution of equations (16) for ρ_1, ρ_2, and ρ_3 is

$$\rho_1 = \frac{\Delta\rho_1}{\Delta}, \quad \rho_2 = \frac{\Delta\rho_2}{\Delta}, \quad \rho_3 = \frac{\Delta\rho_3}{\Delta}, \tag{27}$$

where

$$\Delta\rho_1 = \begin{vmatrix} X_1, & 0, & 0, & f_1, & g_1, & 0, & 0, & 0, & 0 \\ X_2, & \lambda_2, & 0, & 1, & 0, & 0, & 0, & 0, & 0 \\ X_3, & 0, & \lambda_3, & f_3, & g_3, & 0, & 0, & 0, & 0 \\ Y_1, & 0, & 0, & 0, & 0, & f_1, & g_1, & 0, & 0 \\ Y_2, & \mu_2, & 0, & 0, & 0, & 1, & 0, & 0, & 0 \\ Y_3, & 0, & \mu_3, & 0, & 0, & f_3, & g_3, & 0, & 0 \\ Z_1, & 0, & 0, & 0, & 0, & 0, & 0, & f_1, & g_1 \\ Z_2, & \nu_2, & 0, & 0, & 0, & 0, & 0, & 1, & 0 \\ Z_3, & 0, & \nu_3, & 0, & 0, & 0, & 0, & f_3, & g_3 \end{vmatrix}, \tag{28}$$

and there are corresponding expressions for $\Delta\rho_2$ and $\Delta\rho_3$. These determinants are very easy to reduce. Moreover, $\Delta\rho_3$ can be obtained from $\Delta\rho_1$ simply by interchanging the subscripts 1 and 3, and changing the sign, and a corresponding relation exists with $\Delta\rho_2$ if f_2 and g_2 are left in as general symbols until after the reduction. On expanding (28) with respect to the elements of the first column and noting the relations among the coefficients, it is found that

$$\Delta_2\rho_1 = -\begin{vmatrix} X_1, & \lambda_2, & \lambda_3 \\ Y_1, & \mu_2, & \mu_3 \\ Z_1, & \nu_2, & \nu_3 \end{vmatrix} + \frac{(f_1 g_3 - f_3 g_1)}{g_3} \times \begin{vmatrix} X_2, & \lambda_2, & \lambda_3 \\ Y_2, & \mu_2, & \mu_3 \\ Z_2, & \nu_2, & \nu_3 \end{vmatrix} + \frac{g_1}{g_3}\begin{vmatrix} X_3, & \lambda_2, & \lambda_3 \\ Y_3, & \mu_2, & \mu_3 \\ Z_3, & \nu_2, & \nu_3 \end{vmatrix}. \tag{29}$$

The corresponding expressions for ρ_2 and ρ_3 are

$$\left.\begin{aligned} \Delta_2\rho_2 &= +\frac{g_3}{(f_1 g_3 - f_3 g_1)}\begin{vmatrix} \lambda_1, & X_1, & \lambda_3 \\ \mu_1, & Y_1, & \mu_3 \\ \nu_1, & Z_1, & \nu_3 \end{vmatrix} - \begin{vmatrix} \lambda_1, & X_2, & \lambda_3 \\ \mu_1, & Y_2, & \mu_3 \\ \nu_1, & Z_2, & \nu_3 \end{vmatrix} - \frac{g_1}{f_1 g_3 - f_3 g_1}\begin{vmatrix} \lambda_1, & X_3, & \lambda_3 \\ \mu_1, & Y_3, & \mu_3 \\ \nu_1, & Z_3, & \nu_3 \end{vmatrix}, \\ \Delta_2\rho_3 &= +\frac{g_3}{g_1}\begin{vmatrix} \lambda_1, & \lambda_2, & X_1 \\ \mu_1, & \mu_2, & Y_1 \\ \nu_1, & \nu_2, & Z_1 \end{vmatrix} - \frac{(f_1 g_3 - f_3 g_1)}{g_1} \times \begin{vmatrix} \lambda_1, & \lambda_2, & X_2 \\ \mu_1, & \mu_2, & Y_2 \\ \nu_1, & \nu_2, & Z_2 \end{vmatrix} - \begin{vmatrix} \lambda_1, & \lambda_2, & X_3 \\ \mu_1, & \mu_2, & Y_3 \\ \nu_1, & \nu_2, & Z_3 \end{vmatrix}. \end{aligned}\right\} \tag{30}$$

9. Reduction of the Expressions for ρ_1, ρ_2, and ρ_3.

It might be supposed that approximate values of ρ_1, ρ_2, and ρ_3 can be obtained from equations (29) and (30) by using only the first (known) terms of the expansions of the coefficients. It will be shown, however, that this is in general not the case.

Consider first Δ_2, which is a factor of the left members of these equations, and which is defined in (20). The direction cosines λ_i, μ_i, and ν_i can be developed as converging power series of the form

$$\left.\begin{aligned} \lambda_1 &= \lambda_2 + a_1\,\tau_1 + a_2\,\tau_1^2 + \dots, \\ \lambda_3 &= \lambda_2 + a_1\,\tau_3 + a_2\,\tau_3^2 + \dots, \\ \mu_1 &= \mu_2 + b_1\,\tau_1 + b_2\,\tau_1^2 + \dots, \\ \mu_3 &= \mu_2 + b_1\,\tau_3 + b_2\,\tau_3^2 + \dots, \\ \nu_1 &= \nu_2 + c_1\,\tau_1 + c_2\,\tau_1^2 + \dots, \\ \nu_3 &= \nu_2 + c_1\,\tau_3 + c_2\,\tau_3^2 + \dots. \end{aligned}\right\} \tag{31}$$

Since these are the *parametric* equations of the apparent position of the observed body the convergence is not limited by points of inflection, cusps, or loops in the apparent orbit. On making use of these series, the expression for Δ_2 becomes

$$\Delta_2 = \begin{vmatrix} \lambda_2 + a_1\,\tau_1 + a_2\,\tau_1^2 \dots, & \lambda_2, & \lambda_2 + a_1\,\tau_3 + a_2\,\tau_3^2 \dots \\ \mu_2 + b_1\,\tau_1 + b_2\,\tau_1^2 \dots, & \mu_2, & \mu_2 + b_1\,\tau_3 + b_2\,\tau_3^2 \dots \\ \nu_2 + c_1\,\tau_1 + c_2\,\tau_1^2 \dots, & \nu_2, & \nu_2 + c_1\,\tau_3 + c_2\,\tau_3^2 \dots \end{vmatrix}. \tag{32}$$

Substract the second column from each of the others; then τ_1 and τ_3 can be removed as factors from the first and third columns respectively. Substract the resulting first column from the resulting third column. Since $\tau_3^j - \tau_1^j$ is divisible by $\tau_3 - \tau_1$, the resulting third column is divisible by $\tau_3 - \tau_1$; and, as these operations have not changed the value of the determinant, Δ_2 takes the form

$$(33)\qquad \Delta_2 = \tau_1 \tau_3 (\tau_3 - \tau_1) \begin{vmatrix} a_1 + a_2 \tau_1 \ldots , \lambda_2 , a_2 \ldots \\ b_1 + b_2 \tau_1 \ldots , \mu_2 , b_2 \ldots \\ c_1 + c_2 \tau \ldots , \nu_2 , c_2 \ldots \end{vmatrix}$$

$$= \begin{vmatrix} \lambda_1 + \lambda_3 - 2\lambda_2 , \lambda_2 , \lambda_3 - \lambda_2 \\ \mu_1 + \mu_3 - 2\mu_2 , \mu_2 , \mu_3 - \mu_2 \\ \nu_1 + \nu_3 - 2\nu_2 , \nu_2 , \nu_3 - \nu_2 \end{vmatrix} .$$

Therefore Δ_2 is of the third order in the small quantities τ_1, τ_2, and τ.

It will be more convenient in reducing the right members of (29) and (30) to use in place of τ_1 and τ_3 the quantities τ and ϵ, where the latter are defined by

$$(34)\qquad \begin{cases} k(t_3 - t_1) = \tau_3 - \tau_1 = 2\tau , \\ \tau_1 = -\tau + \epsilon , \tau_3 = +\tau + \epsilon . \end{cases}$$

When the successive observations are equidistant ϵ is zero; when they are separated by approximately equal intervals ϵ is small. The latter condition will generally be satisfied in practice. It will be supposed in order to have a guide in making the expansions, that ϵ is of the numerical order of τ^2. In τ and ϵ the expression for Δ_2 becomes

$$(35)\qquad \Delta_2 = -2\tau(\tau^2 - \epsilon^2) \begin{vmatrix} a_1 - a_2\tau \ldots , \lambda_2 , a_2 \ldots \\ b_1 - b_2\tau \ldots , \mu_2 , b_2 \ldots \\ c_1 - c_2\tau \ldots \nu_2 , c_2 \ldots \end{vmatrix} ,$$

Therefore Δ_2 is expansible as a power series in τ and ϵ and the term of lowest degree in this expansion is of the third order.

Consider the right member of the first equation of (30). Both the coefficients of the determinants and also the determinants can be expanded as converging power series in τ and ϵ. Therefore the right members can be developed as power series in τ and ϵ. Since the left member starts with a term of the third order, the right member will also start with a term of the third order. It will be shown that the terms of lowest (third) order in the right member depend not only upon the first terms of the expansions of the coefficients, which are known, but also upon the second terms, which involve the unknown $u = 1/r_2^3$. For this reason the second terms may not be neglected in making the first approximation to the solution.

It follows from equations (15) and (34) that

$$(36)\qquad \begin{cases} \dfrac{+g_3}{f_1 g_3 - f_3 g_1} = \dfrac{1}{2} + \dfrac{\epsilon}{2\tau} + \dfrac{u\tau^2}{4} P + \dfrac{u\tau\epsilon}{12} Q , \\[2ex] \dfrac{-g_1}{f_1 g_3 - f_3 g_1} = \dfrac{1}{2} - \dfrac{\epsilon}{2\tau} + \dfrac{u\tau^2}{4} P - \dfrac{u\tau\epsilon}{12} Q , \\[2ex] P = 1 - \dfrac{\epsilon^2}{\tau^2} - 2p\,\epsilon \\ \qquad + \frac{1}{12}(7u - 15p^2 + 3q)\,\tau^2 \ldots , \\[2ex] Q = 1 - \dfrac{\epsilon^2}{\tau^2} + \frac{3}{2}p\,\dfrac{\tau^2}{\epsilon} - 3p\epsilon + \frac{1}{60}(37u \\ \quad - 765p^2 + 153q)\,\tau^2 + \frac{1}{4}p(3u + 14p^2 - 6q)\,\dfrac{\tau^4}{\epsilon} \ldots , \end{cases}$$

where all terms up to the sixth order have been written. On making use of these developments the expression for $\Delta_2\rho_2$ becomes

$$\Delta_2\rho_2 = \tfrac{1}{2}K + \frac{\epsilon}{2\tau} K_1 + \frac{u\tau^2}{4} PK_2 + \frac{u\tau\epsilon}{12} QK_1 , \qquad (37)$$

where

$$(38)\qquad \begin{cases} K = \begin{vmatrix} \lambda_1 , X_1 , \lambda_3 \\ \mu_1 , Y_1 , \mu_3 \\ \nu_1 , Z_1 , \nu_3 \end{vmatrix} - 2 \begin{vmatrix} \lambda_1 , X_2 , \lambda_3 \\ \mu_1 , Y_2 , \mu_3 \\ \nu_1 , Z_2 , \nu_3 \end{vmatrix} + \begin{vmatrix} \lambda_1 , X_3 , \lambda_3 \\ \mu_1 , Y_3 , \mu_3 \\ \nu_1 , Z_3 , \nu_3 \end{vmatrix} , \\[2ex] K_1 = \begin{vmatrix} \lambda_1 , X_1 , \lambda_3 \\ \mu_1 , Y_1 , \mu_3 \\ \nu_1 , Z_1 , \nu_3 \end{vmatrix} - \begin{vmatrix} \lambda_1 , X_3 , \lambda_3 \\ \mu_1 , Y_3 , \mu_3 \\ \nu_1 , Z_3 , \nu_3 \end{vmatrix} , \\[2ex] K_2 = \begin{vmatrix} \lambda_1 , X_1 , \lambda_3 \\ \mu_1 , Y_1 , \mu_3 \\ \nu_1 , Z_1 , \nu_3 \end{vmatrix} + \begin{vmatrix} \lambda_1 , X_3 , \lambda_3 \\ \mu_1 , Y_3 , \mu_3 \\ \nu_1 , Z_3 , \nu_3 \end{vmatrix} . \end{cases}$$

Since these determinants have two columns in common, they can be added giving

$$(39)\qquad \begin{cases} K = \begin{vmatrix} \lambda_1 , X_1 + X_3 - 2X_2 , \lambda_3 - \lambda_1 \\ \mu_1 , Y_1 + Y_3 - 2Y_2 , \mu_3 - \mu_1 \\ \nu_1 , Z_1 + Z_3 - 2Z_2 , \nu_3 - \nu_1 \end{vmatrix} , \\[2ex] K_1 = \begin{vmatrix} \lambda_1 , X_1 - X_3 , \lambda_3 - \lambda_1 \\ \mu_1 , Y_1 - Y_3 , \mu_3 - \mu_1 \\ \nu_1 , Z_1 - Z_3 , \nu_3 - \nu_1 \end{vmatrix} , \\[2ex] K_2 = \begin{vmatrix} \lambda_1 , X_1 + X_3 , \lambda_3 - \lambda_1 \\ \mu_1 , Y_1 + Y_3 , \mu_3 - \mu_1 \\ \nu_1 , Z_1 + Z_3 , \nu_3 - \nu_1 \end{vmatrix} . \end{cases}$$

The quantities $X_1 , X_3 , \ldots$ can be expanded as power series in τ and ϵ of the form

$$(40)\qquad \begin{cases} X_1 = X_2 + A_1(-\tau + \epsilon) + A_2(-\tau + \epsilon)^2 + \ldots , \\ X_3 = X_2 + A_1(\tau + \epsilon) + A_2(\tau + \epsilon)^2 + \ldots . \end{cases}$$

Therefore the second column of the expression for K is of the second order, and the third column is of the first order. *Therefore K, which is the first known term in the right member of the expression for $\Delta_2\rho_2$, is of the third order.* The determinant K_1 is of the second order, and in the known part of the right member it is multiplied by a term of the first order. Therefore the known part of the right member of the expression for $\Delta_2\rho_2$ is of the third order. Since K_2 is of the first order, while P_2 is of order zero, the third term of the expression for $\Delta_2\rho_2$, which involves the unknown $u = 1/r_2^3$, is also of the third order and must be retained even in the first approximation. The last term of the right member of (37) is of the fifth order and may be neglected in a first approximation, and sometimes altogether.

In order to put the expressions for $\Delta_2\rho_1$ and $\Delta_2\rho_3$ in such a form that they can be computed accurately with four or five-place tables let

$$
(41)\left\{
\begin{aligned}
\frac{f_1 g_3 - f_3 g_1}{g_3} &= \frac{2}{1+\frac{\epsilon}{\tau}+\frac{1}{2}u\tau(\tau P+\frac{1}{3}\epsilon Q)} \\
&= 2 - 2\frac{\epsilon}{\tau} + F_1\,, \\
\frac{f_1 g_3 - f_3 g_1}{-g_1} &= \frac{2}{1-\frac{\epsilon}{\tau}+\frac{1}{2}u\tau(\tau P-\frac{1}{3}\epsilon Q)} \\
&= 2 + \frac{2\epsilon}{\tau} + F_3\,, \\
\frac{g_1}{g_3} &= \frac{-\left[1-\frac{\epsilon}{\tau}+\frac{1}{2}u\tau(\tau P-\frac{1}{3}\epsilon Q)\right]}{\left[1+\frac{\epsilon}{\tau}+\frac{1}{2}u\tau(\tau P+\frac{1}{3}\epsilon Q)\right]} \\
&= -1 + \frac{2\epsilon}{\tau} + G_1\,, \\
\frac{g_3}{g_1} &= \frac{-\left[1+\frac{\epsilon}{\tau}+\frac{1}{2}u\tau(\tau P+\frac{1}{3}\epsilon Q)\right]}{\left[1-\frac{\epsilon}{\tau}+\frac{1}{2}u\tau(\tau P-\frac{1}{3}\epsilon Q)\right]} \\
&= -1 - \frac{2\epsilon}{\tau} + G_3
\end{aligned}
\right.
$$

where

$$
(42)\left\{
\begin{aligned}
F_1 &= \frac{2\frac{\epsilon^2}{\tau^2} - u\tau\left(1-\frac{\epsilon}{\tau}\right)(\tau P+\frac{1}{3}\epsilon Q)}{1+\frac{\epsilon}{\tau}+\frac{1}{2}u\tau(\tau P+\frac{1}{3}\epsilon Q)}\,, \\
F_3 &= \frac{2\frac{\epsilon^2}{\tau^2} - u\tau\left(1+\frac{\epsilon}{\tau}\right)(\tau P-\frac{1}{3}\epsilon Q)}{1-\frac{\epsilon}{\tau}+\frac{1}{2}u\tau(\tau P-\frac{1}{3}\epsilon Q)}\,, \\
G_1 &= \frac{-2\frac{\epsilon^2}{\tau^2} + \frac{1}{3}u\tau\epsilon Q - u\epsilon(\tau P+\frac{1}{3}\epsilon Q)}{1+\frac{\epsilon}{\tau}+\frac{1}{2}u\tau(\tau P+\frac{1}{3}\epsilon Q)}\,, \\
G_3 &= \frac{-2\frac{\epsilon^2}{\tau^2} - \frac{1}{3}u\tau\epsilon Q + u\epsilon(\tau P-\frac{1}{3}\epsilon Q)}{1-\frac{\epsilon}{\tau}+\frac{1}{2}u\tau(\tau P-\frac{1}{3}\epsilon Q)}\,.
\end{aligned}
\right.
$$

These functions are all of the second order and G_1 and G_3 vanish with ϵ. Therefore when the intervals between the successive observations are very nearly equal, the terms which involve G_1 and G_3 will generally be insensible.

With the developments of (41) the expressions for $\Delta_2\rho_1$ and $\Delta_2\rho_3$ [equations (29) and (30)] become

$$
\left.
\begin{aligned}
\Delta_2\rho_1 &= K_3 + 2\frac{\epsilon}{\tau}K_4 + F_1K_5 + G_1(K_4+K_5)\,, \\
\Delta_2\rho_3 &= K_6 + 2\frac{\epsilon}{\tau}K_7 + F_3K_8 + G_3(K_8-K_7), \\
K_3 &= -\begin{vmatrix} X_1+X_3-2X_2\,, & \lambda_2\,, & \lambda_3-\lambda_2 \\ Y_1+Y_3-2Y_2\,, & \mu_2\,, & \mu_3-\mu_2 \\ Z_1+Z_3-2Z_2\,, & \nu_2\,, & \nu_3-\nu_2 \end{vmatrix}, \\
K_4 &= \begin{vmatrix} X_3-X_2\,, & \lambda_2\,, & \lambda_3-\lambda_2 \\ Y_3-Y_2\,, & \mu_2\,, & \mu_3-\mu_2 \\ Z_3-Z_2\,, & \nu_2\,, & \nu_3-\nu_2 \end{vmatrix}, \\
K_5 &= \begin{vmatrix} X_2\,, & \lambda_2\,, & \lambda_3-\lambda_2 \\ Y_2\,, & \mu_2\,, & \mu_3-\mu_2 \\ Z_2\,, & \nu_2\,, & \nu_3-\nu_2 \end{vmatrix}, \\
K_6 &= -\begin{vmatrix} X_1+X_3-2X_2\,, & \lambda_1\,, & \lambda_2-\lambda_1 \\ Y_1+Y_3-2Y_2\,, & \mu_1\,, & \mu_2-\mu_1 \\ Z_1+Z_3-2Z_2\,, & \nu_1\,, & \nu_2-\nu_1 \end{vmatrix}, \\
K_7 &= \begin{vmatrix} X_2-X_1\,, & \lambda_1\,, & \lambda_2-\lambda_1 \\ Y_2-Y_1\,, & \mu_1\,, & \mu_2-\mu_1 \\ Z_2-Z_1\,, & \nu_1\,, & \nu_2-\nu_1 \end{vmatrix}, \\
K_8 &= \begin{vmatrix} X_2\,, & \lambda_1\,, & \lambda_2-\lambda_1 \\ Y_2\,, & \mu_1\,, & \mu_2-\mu_1 \\ Z_2\,, & \nu_1\,, & \nu_2-\nu_1 \end{vmatrix}.
\end{aligned}
\right\} \quad (43)
$$

After u has been found these equations approximately determine ρ_1 and ρ_3; and the determination is exact when u, p, and q are known.

10. Approximate Determination of ρ_2 and r_2.

Since $u = 1/r_2^3$, equation (37) can be written in the form

$$
\left.
\begin{aligned}
\Delta_2\rho_2 &= \tfrac{1}{2}K + \frac{\epsilon}{2\tau}K_1 + \frac{\Phi}{r_2^3}, \\
\Phi &= \frac{\tau^2}{4}P\,K_2 + \frac{\tau\epsilon}{12}Q\,K_1 \\
&= \frac{\tau}{4}\left(1-\frac{\epsilon^2}{\tau^2}\right)\left[\tau K_2 + \frac{\epsilon}{3}K_1\right] + \Psi\,,
\end{aligned}
\right\} \quad (44)
$$

where Ψ includes the unknown terms of higher order which may be neglected in the first approximation. Generally ϵ^2/τ^2 may also be neglected at first.

Let R_2 represent the distance from the observer to the Sun, ψ the known angle at the observer between ρ_2 and R_2, and φ the unknown angle at the observed body between ρ_2 and r_2. Then it follows from the geometrical relations that

$$
(45) \quad \left\{ \begin{aligned} R_2 \cos\psi &= -\lambda_2 X_2 - \mu_2 Y_2 - \nu_2 Z_2 \,, \\ \rho_2 &= R_2 \frac{\sin(\psi+\varphi)}{\sin\varphi} \,, \\ r_2 &= R_2 \frac{\sin\psi}{\sin\varphi} \,. \end{aligned} \right.
$$

Substitute these expressions for ρ_2 and r_2 in (44) and let

$$
(46) \quad \left\{ \begin{aligned} N \sin m &= R_2 \sin\psi \,, \\ N \cos m &= R_2 \cos\psi - \frac{K}{2\Delta_2} - \frac{\epsilon K_1}{2\tau\,\Delta_2} \,, \\ M &= \frac{N\,\Delta_2 R_2^3 \sin^3\psi}{\Phi} \,, \end{aligned} \right.
$$

where the sign of N is taken so that M shall be positive. Then the equation which defines φ is

$$
(47) \qquad \sin^4\varphi = M \sin(\varphi + m) \,,
$$

where m and M are known, at least except for the terms of higher order in Φ which must be neglected in a first approximation.

Equation (47) is encountered in one form or another in all general methods of determining orbits. It was exhaustively discussed by BENJAMIN PEIRCE, especially from the geometric point of view, and his results are included in the appendix (pp. 308–313) for DAVIS' edition of GAUSS' *Theoria Motus*.* It has either one or two admissible solutions according to the position of the observed body.†

It follows from (36), (37), and (44) that the first unknown term in Φ is $-\frac{1}{3}up\tau^3 K_1$, and since this must be neglected in the first approximation it determines the order of the error in the result. Since K_1 is of the first order this term is of the fourth order. And since p is proportional to the derivative of r^2 with respect to τ, by (14), it carries the eccentricity of the orbit of the observed body as a factor. For this reason, if the observed body is a small planet, the term in question is really of the fifth order. The numerator of the expression for M, equations (46), is of the third order because of the factor Δ_2, and the denominator Φ is of the third order. Hence the error in M is of the first order, or of the second order if p is supposed to be of the first order. The error in φ is in general of the same order as the error in M. The most unfavorable case is that in which there are two roots of (47) in the neighborhood of $\pi/2$. An error of the first order in φ introduces an error of the first order in ρ_2 and r_2, equations (45); and the errors in ρ_1 and ρ_3, equations (43), are of the same order.

* The second method given there seems to be identical, except for differences of notation, with the method of BERNSTEIN. Note by LEUSCHNER and BERNSTEIN, *Bulletin of Am. Math. Soc.*, Vol. XVIII, pp. 168–9.

† CHARLIER, *Meddelande fran Lunds Observatorium*, No. 45.

11. THE SPECIAL CASE $\Delta_2 = 0$.

It was shown in section 5 that Δ_2 is distinct from zero unless the three apparent positions of the observed body lie on the arc of a great circle. Suppose Δ_2 is zero and consider the problem of determining r_2, ρ_1, ρ_2, and ρ_3.

If K and the first two terms of Φ are large compared to Ψ the value of r_2 is determined from the first equation of (44), and then ρ_2 is given by the second of (45). In order to obtain ρ_1 and ρ_3 equations which correspond to these can be developed taking t_1 and t_3 respectively, instead of t_2, as origins of time.

There is a special sub-case in which both K and Φ vanish, viz., that in which the apparent positions of the observed body are the same at t_1 and t_3, because under these conditions the third columns of Δ_2, K, K_1, and K_2 are all zero. But even in this case the problem is in general not indeterminate. Each of the determinants contains as a factor the sine of the angle between ρ_1 and ρ_3, as can easily be seen by taking the plane through the positions of the observed body at t_1 and t_3 as the xy-plane, and directing the x-axis to the position at t_1. After this factor is removed the determinants will not in general be zero, and r_2 can be determined as before. If they are zero, the problem is indeterminate because the determinants contain no other common factor. If the observed position at t_2 is distinct from those at t_1 and t_3 the distances ρ_1 and ρ_3 are found just as when all three positions are distinct. If the observed position at t_2 is identical with that at t_1 or t_3 the corresponding factor must be removed from the equation where its value is zero. If the observed positions at all three epochs are the same, the zero factor must be removed from each of the equations corresponding to the first of (44). In general, it is not only not fatal if the three observed positions lie on a great circle, but they may even be identical.

12. REDUCTION OF THE TIME.

There are three methods of making correction for the time required for light to pass from the observed body to the observer which were clearly set forth by GAUSS in the *Theoria Motus*, Arts. 71 and 118. The third method is preferable to the others for present purposes, because in it only the times of the observations are changed. Since the observed data and the coördinates of the Earth are unaltered it will not be necessary to recompute any of the determinants.

Light travels an astronomical unit in 498.65 seconds. Therefore if the distances ρ_1, ρ_2, and ρ_3 are expressed in astronomical units, the corrected times of the observations are

$$t_1 - 498^s.65\,\rho_1\,, \qquad t_2 - 498^s.65\,\rho_2\,, \qquad t_3 - 498^s.65\,\rho_3\,.$$

With these corrected values of the epochs of the observations new values of τ and ϵ are to be computed. The subsequent changes in ρ_1, ρ_2, and ρ_3 in general will be so small that the time will require no further corrections.

13. Computation of x_2, y_2, and z_2.

Since g_2 is zero the second, fifth and eighth equations of (16) give respectively

$$(48) \qquad \left\{\begin{aligned} x_2 &= \lambda_2\rho_2 + X_2\,, \\ y_2 &= \mu_2\rho_2 + Y_2\,, \\ z_2 &= \nu_2\rho_2 + Z_2\,. \end{aligned}\right.$$

These equations define x_2, y_2, and z_2 with the same order of accuracy as that to which ρ_2 is known. But it should be noted that though ρ_2 may not have been determined to many places, or even if it is not possible to compute it to many places because Δ_2 is small, yet x_2, y_2, and z_2 may be determined so that all three equations are satisfied to the number of places given by the observations.

14. Computation of x_2', y_2', and z_2'.

The first and third equations of (16) give

$$\left\{\begin{aligned} g_1\,x_2' &= X_1 - f_1\,x_2 + \lambda_1\,\rho_1\,, \\ g_3\,x_2' &= X_3 - f_3\,x_2 + \lambda_3\,\rho_3\,, \end{aligned}\right.$$

either of which uniquely defines x_2'. There are corresponding equations for y_2' and z_2'. In general the two values of x_2', y_2' and z_2' will not be exactly the same, especially in the first approximation. If they fail to agree rather closely the approximation is not sufficiently exact. If x_2', y_2', and z_2' are determined by the first equation of (49) and the corresponding ones for y_2' and z_2' so as exactly to satisfy the equations to the number of places given by the observations, then the elements based on them and the x_2, y_2, and z_2, defined by (48), will exactly represent the observations at t_1 and t_2. But there will be differences at t_3 unless the values of x_2', y_2', and z_2' determined by the third, sixth, and ninth equations of (16) are the same as those which were used.

In the first approximation it is advantageous to use the average of the two values of x_2' determined by (49), or what is essentially the same thing (including y_2' and z_2')

$$(49) \qquad \left\{\begin{aligned} (g_3 - g_1)x_2' &= X_3 - X_1 - (f_3 - f_1)\,x_2 + \lambda_3\rho_3 - \lambda_1\rho_1\,, \\ (g_3 - g_1)y_2' &= Y_3 - Y_1 - (f_3 - f_1)\,y_2 + \mu_3\rho_3 - \mu_1\rho_1\,, \\ (g_3 - g_1)z_2' &= Z_3 - Z_1 - (f_3 - f_1)\,z_2 + \nu_3\rho_3 - \nu_1\rho_1\,. \end{aligned}\right.$$

It is found from equations (15) and (34) that

$$(50) \qquad \left\{\begin{aligned} g_3 - g_1 &= 2\tau[1 - \tfrac{1}{6}u\,\tau^2 - \tfrac{1}{2}u\epsilon^2 + u\,p\,\tau^2\,\epsilon \\ &\qquad + \tfrac{1}{120}u\,(u - 45p^2 + 9q)\,\tau^4 \ldots]\,, \\ f_3 - f_1 &= 2\tau[-u\epsilon + \tfrac{1}{2}up\tau^2 + \tfrac{3}{2}up\epsilon^2 + \tfrac{1}{6}u(u - 15p^2 + 3q)\tau^2\epsilon \\ &\qquad - \tfrac{1}{8}u\,p\,(u - 7p^2 + 3q)\,\tau^4 + \ldots]\,. \end{aligned}\right.$$

The errors in x_2, y_2, z_2, u, ρ_1, ρ_2, and ρ_3 are of the first and second orders respectively according as the eccentricity is large or near zero. It is important to find whether or not x_2', y_2', and z_2' are determined by (49) with the same order of accuracy. The difference $g_3 - g_1$ is of the first order and $f_3 - f_1$ is of the third order. Therefore the error in x_2' coming from $(f_3 - f_1)\,x_2$ is two orders higher than that in x_2, and no difficulty arises from this term.

The errors coming from the terms $\lambda_3\rho_3$ and $\lambda_1\rho_1$ are one order lower respectively than those in ρ_3 and ρ_1. It will be shown that in the difference $\lambda_3\rho_3 - \lambda_1\rho_1$ the error of lowest order cancels out so that the order of accuracy of x_2' is actually the same as that of x_2, ρ_1, ρ_3. Inasmuch as $\lambda_3 = \lambda_1 + (\lambda_3 - \lambda_1)$, it follows that

$$\lambda_3\rho_3 - \lambda_1\rho_1 = \lambda_1(\rho_3 - \rho_1) + (\lambda_3 - \lambda_1)\rho_3\,.$$

The difference $\lambda_3 - \lambda_1$ is of the first order, and therefore the error coming from $(\lambda_3 - \lambda_1)\rho_3$ is of the same order as that of ρ_3. It follows from (43) that

$$\begin{aligned} \Delta_2\,(\rho_3 - \rho_1) = K_6 - K_3 + \frac{2\epsilon}{\tau}(K_7 - K_4) + F_3\,K_8 \\ - F_1K_5 + G_3(K_8 - K_7) + G_1(K_4 + K_5)\,. \end{aligned}$$

The only errors to which the right member of this equation is subject are in F_1, F_3, G_1, and G_3. The terms of lowest order in these quantities are seen from (36) and (42) to be respectively $-u\tau^2$, $-u\tau^2$, $-\frac{2}{3}u\tau\epsilon + \frac{1}{2}up\tau^3$, and $+\frac{2}{3}u\tau\epsilon - \frac{1}{2}up\tau^3$. Hence it is found, on making use of the expressions for K_3, ..., K_8 in (43), that the term of the lowest order in the right member of $\Delta_2(\rho_3 - \rho_1)$ which is subject to the errors of the first approximation is

$$-u\tau^2 \begin{vmatrix} X_2\,, & \lambda_2\,, & \lambda_1 + \lambda_3 - 2\lambda_2 \\ Y_2\,, & \mu_2\,, & \mu_1 + \mu_3 - 2\mu_2 \\ Z_2\,, & \nu_2\,, & \nu_1 + \nu_3 - 2\nu_2 \end{vmatrix}\,.$$

This term is of the fourth order and all other terms are at least of the fifth order. Since Δ_2 is of the third order the coefficient of u in the expression for $\lambda_1(\rho_3 - \rho_1)$ is of the first order. Hence the error in x_2', so far as it depends on the error in $\lambda_3\rho_3 - \lambda_1\rho_1$, is of the same order as the error in u.

15. The Second and Higher Approximations.

Suppose x_2, y_2, z_2, x_2', y_2', and z_2' have been determined in the first approximation up to small quantities of the first order. Then $u = 1/r_2^3$, p, and q can be com-

puted from (11) and (14) to the same order of accuracy. Also P can be determined up to the third order and Q up to the first order by equations (36). Therefore Φ, equation (44), is determined up to terms of the sixth order, or one higher than in the first approximation. Therefore the second approximation is actually more accurate than the first, and the process may be continued until all terms of any desired order are included.

The method of approximation of GAUSS, depending on the ratios of the triangles between the heliocentric radii to the corresponding sectors, can also be used because it follows from equations (13) that

$$(51)\qquad \left\{\begin{aligned} -g_1 &= \frac{(x_1 y_2 - x_2 y_1)}{x_0 y_0' - x_0' y_0}\,,\\ g_3 &= \frac{(x_2 y_3 - x_3 y_2)}{x_0 y_0' - x_0' y_0}\,,\\ f_1 g_3 - f_3 g_1 &= \frac{(x_1 y_3 - x_3 y_1)}{x_0 y_0' - x_0' y_0}\,.\end{aligned}\right.$$

The denominators are twice the areal velocity of the body in its orbit, and therefore g_1, g_3, and $f_1 g_3 - f_3 g_1$ divided by τ_1, τ_3 and 2τ are the ratios of GAUSS. It was of course to be expected that such relationships should appear since all methods involve the same fundamentals.

Suppose the successive approximations have been carried so far that all sensible terms in the series have been included. The question is whether or not the theoretical positions will agree with the observed. It should be noted that x_2, y_2, z_2, x_2', y_2', and z_2' are equivalent to the six elements of the orbit, which are numerically well determined from them. The series f_i and g_i furnish a direct method of computing theoretical positions which are equivalent to the ordinary elements for short enough intervals of time. Consequently the whole question reduces to that of finding whether equations (16) are satisfied.

It follows from the method of determining x_2, y_2, and z_2 that the observations at t_2 will be exactly satisfied even in the first approximation. Each of the quantities x_2', y_2', and z_2' was determined from the difference of two equations. Consequently these equations will not necessarily be exactly satisfied; but if it is desired x_2', y_2', and z_2' can be so determined that one of each pair will be fulfilled. With this determination of the arbitraries the theoretical and observed positions will agree at two of the dates of observation, and in general the correspondence will be very close for the third because they were involved symmetrically in the computations down to the last step. In the method of GAUSS the calculations are arranged so that the first and third positions are satisfied, and the test of the orbit is the comparison of theory and observation at the second date. In the method of LAPLACE the geocentric distance and its derivative at the second date are computed from determinants, and then the coördinates and velocities at the second date are determined from the geometric relations connecting the Sun, observer, and observed body and the derivatives of these relations. It follows that the derived elements need satisfy only the second observation. This is the reason the method of LAPLACE has given unsatisfactory results in practice. It does not mean that the elements found by it are necessarily much in error; they may be in fact as nearly correct as those which satisfy the observations. The question is analogous to that of solving equations (1). If y and z are found from the determinants and x is then computed from one of the equations, the other two will not in general be exactly satisfied. If only z is computed by determinants and x and y are then determined from two of the equations, these two will be exactly satisfied unless, indeed, it happens that the determinant of the coefficients of x and y in them is small. Moreover, the third equation will in general be more nearly satisfied than when y and z are computed by determinants. But, as was explained in discussing equations (1), the fact that they are exactly or nearly satisfied does not prove that the solution is very exact when their determinant is small.

In LEUSCHNER's modification of the method of HARZER, which has for its basis the method of LAPLACE, four of the six quantities are determined by successive differential corrections so that the first and third observations shall be satisfied. The second observation in general is not exactly satisfied, but adjustments can be made so that it will be fulfilled. This procedure brings theory and observation into harmony, whereas the original method of LAPLACE does not in general do so. But when the intervals between the observations are small, so that Δ_2 is small, the advantage of LEUSCHNER's procedure as compared with that of LAPLACE is only apparent, because the elements are partially indeterminate as was explained in discussing equations (1); and the fulfillment of the observations is no test of the accuracy of the results. The three sets of elements quoted in the introduction sufficiently illustrate the point.

If it is desired small corrections to the approximate values of ρ_1, ρ_2, ρ_3, x_2, y_2, z_2, x_2', y_2', and z_2' can be determined so that all of equations (16) will be satisfied. The only difficulty arises when the Jacobian of the function of each line with respect to these nine quantities is very small. This happens only exceptionally, as will be shown in the next section. Consequently there is in general no difficulty, except the labor of making the differential corrections, in deriving elements which will exactly satisfy all three observations. But to make the adjustments is justifiable neither theoretically nor practically, because the accuracy is only apparent, and even if it were not so

it would not be required by observers. The place for the fine adjustments is in the definitive determination of the orbit based on all valuable observational data.

16. A General Analytic Solution.

The method of approximation to the solution of analytic functions which has been used is closely related to that of direct power series developments, and the two processes are valid under the same conditions. In the present case some preliminary transformations and considerations are necessary to bring the problem directly under the general theory of implicit functions; probably this is the reason that the problem has been treated this way only by Lagrange.* The method of Lagrange has been completed by Charlier in *Meddelande från Lunds Observatorium*, No. 46.

Suppose equations (16) considered as a linear system in $\rho_1, \rho_2, \ldots, x_2'$ are solved for these quantities. After the solution let the f_i and g_i be expressed in terms of x_2, y_2, z_2, x_2', y_2', and z_2'. The result will have the form

$$(52)\qquad \left\{\begin{aligned} \Delta_2\,\rho_1 &= \varphi_1(x_2, y_2, z_2, x_2', y_2', z_2'),\\ &\;\vdots\\ \Delta_2\,x_2 &= \varphi_4(x_2, y_2, z_2, x_2', y_2', z_2'),\\ &\;\vdots\\ \Delta_2\,z_2' &= \varphi_9(x_2, y_2, z_2, x_2', y_2', z_2'). \end{aligned}\right.$$

It was shown in section 9 that Δ_2 is of the third order in τ_1, τ_2, and τ_3, or in τ and $\sqrt{\epsilon}$. It is convenient for present purposes to define $c, a_1', a_2', b_1', \ldots$ by

$$(53)\qquad \left\{\begin{aligned} \epsilon &= c\,\tau^2,\\ \lambda_1 &= \lambda_2 - a_1'\tau(1 - c\,\tau),\\ \lambda_3 &= \lambda_2 + a_1'\tau(1 + c\,\tau) - a_2'\tau^2(1 + c\,\tau),\\ \mu_1 &= \mu_2 - b_1'\tau(1 - c\,\tau),\\ \mu_3 &= \mu_2 + b_1'\tau(1 + c\,\tau) - b_2'\tau^2(1 + c\,\tau),\\ \nu_1 &= \nu_2 - c_1'\tau(1 - c\,\tau),\\ \nu_3 &= \nu_2 + c_1'\tau(1 + c\,\tau) - c_2'\tau^2(1 + c\,\tau). \end{aligned}\right.$$

The constants $c, a_1', a_2', \ldots$ are uniquely determined and they are of order zero. On using these expressions for $\lambda_1, \lambda_3, \ldots$ the determinant Δ_2 becomes

$$(54)\qquad \Delta_2 = \tau^3(1-c^2\tau^2)\begin{vmatrix} a_1', & \lambda_2, & a_2'\\ b_1', & \mu_2, & b_2'\\ c_1', & \nu_2, & c_2' \end{vmatrix} = \tau^3(1-c^2\tau^2)\Delta_2^{(0)}.$$

Since Δ_2 is of the third order in τ the determinant $\Delta_2^{(0)}$ is of order zero.

*Collected Works, Vol. IV, p. 496–532.

Suppose all the determinants entering the right members of equations (52), which involve $X_1, Y_1, Z_1, \ldots$, are developed in a similar manner. Since $\rho_1, \ldots, z_2'$ are of order zero in τ, that is since they do not approach zero as a limit as the intervals between the successive observations approach zero as a limit, the right members are divisible by τ^3. It was shown in (37) and (43) that the terms of lowest (third) order in the expressions for $\Delta_2\rho_1$, $\Delta_2\rho_2$, and $\Delta_2\,\rho_3$ do not involve x_2', y_2', and z_2'. It is true also of the expressions for $\Delta_2 x_2, \ldots, \Delta_2 z_2'$. Consequently equations (52) can be written in the form

$$\left.\begin{aligned} \Delta_2^{(0)}\rho_1 - \varphi_1^{(0)}(x_2, y_2, z_2) &= \varphi_1^{(1)}\tau + \ldots,\\ &\;\vdots\\ \Delta_2^{(0)}x_2 - \varphi_4^{(0)}(x_2, y_2, z_2) &= \varphi_4^{(1)}\tau + \ldots,\\ &\;\vdots\\ \Delta_2^{(0)}z_2' - \varphi_9^{(0)}(x_2, y_2, z_2) &= \varphi_9^{(1)}\tau + \ldots, \end{aligned}\right\}\qquad(55)$$

where $\varphi_1^{(1)}, \ldots, \varphi_9^{(1)}$ are regular analytical functions of $x_2, y_2, z_2, x_2', y_2', z_2'$, and where the right members are infinite power series in τ.

The solution of equations (55) is only a problem of implicit functions. Suppose that for $\tau = 0$ a solution of (55) is

$$\left.\begin{aligned} \rho_1 &= \rho_1^{(0)}, & \rho_2 &= \rho_2^{(0)}, & \rho_3 &= \rho_3^{(0)},\\ x_2 &= x_2^{(0)}, & y_2 &= y_2^{(0)}, & z_2 &= z_2^{(0)},\\ x_2' &= x_2^{(0)\prime}, & y_2 &= y_2^{(0)\prime}, & z_2 &= z_2^{(0)\prime}. \end{aligned}\right\}\qquad(56)$$

Then the corresponding general solution has the form

$$\left.\begin{aligned} \rho_1 &= \rho_1^{(0)} + \rho_1^{(1)}\tau + \rho_1^{(2)}\tau^2 + \ldots,\\ &\;\vdots\\ z_2' &= z_2^{(0)\prime} + z_2^{(1)\prime} + z_2^{(2)\prime}\tau^2 + \ldots \end{aligned}\right\}\qquad(57)$$

provided the Jacobian of the left members of (55) with respect to $\rho_1, \rho_2, \ldots, z_2'$ is distinct from zero for $\rho_1 = \rho_1^{(0)}$, $\rho_2 = \rho_2^{(0)}, \ldots, z_2' = z_2'^{(0)}$. If the Jacobian is zero the solution in general is expansible as a power series in some fractional power of τ, the fraction depending upon supplementary conditions. In all cases the solution exists either in integral or fractional powers of τ. The series really involve two variables, τ and c. In general the case of two independent parameters presents difficulties when the Jacobian vanishes, but in the present case the convergence is not imperiled by small values of c because c occurs only where it is multiplied by some positive power of τ.

Since $\rho_1, \rho_2, \rho_3, x_2', y_2'$, and z_2' each appear but once and to the first degree in the left members of (55), the Jacobian reduces to

$$\frac{J}{(\Delta_2^{(0)})^6} = \begin{vmatrix} \Delta_2^{(0)} - \dfrac{\partial \varphi_4^{(0)}}{\partial x_2}, & -\dfrac{\partial \varphi_4^{(0)}}{\partial y_2}, & -\dfrac{\partial \varphi_4^{(0)}}{\partial z_2} \\ -\dfrac{\partial \varphi_5^{(0)}}{\partial x_2}, & \Delta_2^{(0)} - \dfrac{\partial \varphi_5^{(0)}}{\partial y_2}, & -\dfrac{\partial \varphi_5^{(0)}}{\partial z_2} \\ -\dfrac{\partial \varphi_6^{(0)}}{\partial x_2}, & -\dfrac{\partial \varphi_6^{(0)}}{\partial y_2}, & \Delta_2^{(0)} - \dfrac{\partial \varphi_6^{(0)}}{\partial z_2} \end{vmatrix}.$$

It follows from the second, fifth, and eighth equations of (16) that

$$\varphi_4^{(0)} = \lambda_2 \varphi_2^{(0)}, \quad \varphi_5^{(0)} = \mu_2 \varphi_2^{(0)}, \quad \varphi_6^{(0)} = \nu_2 \varphi_2^{(0)}.$$

The quantity $\varphi_2^{(0)}$ is needed only so far as it involves x_2, y_2, and z_2. It is seen from equations (37) and (39) that so far as $\varphi_2^{(0)}$ involves x_2, y_2, and z_2 its value is

$$\varphi_2^{(0)} = \frac{K_2^{(0)}}{4 r_2^3},$$

where $K_2^{(0)}$ is the part of K_2 of order zero which remains after τ^2 has been divided out.

The determinant $J/(\Delta_2^{(0)})^6$ therefore becomes

$$\begin{vmatrix} \Delta_2^{(0)} + \dfrac{3\lambda_2 K_2^{(0)} x_2}{4 r_2^5}, & +\dfrac{3\lambda_2 K_2^{(0)} y_2}{4 r_2^5}, & +\dfrac{3\lambda_2 K_2^{(0)} z_2}{4 r_2^5} \\ +\dfrac{3\mu_2 K_2^{(0)} x_2}{4 r_2^5}, & \Delta_2^{(0)} + \dfrac{3\mu_2 K_2^{(0)} y_2}{4 r_2^5}, & +\dfrac{3\mu_2 K_2^{(0)} z_2}{4 r_2^5} \\ +\dfrac{3\nu_2 K_2^{(0)} x_2}{4 r_2^5}, & +\dfrac{3\nu_2 K_2^{(0)} y_2}{4 r_2^5}, & \Delta_2^{(0)} + \dfrac{3\nu_2 K_2^{(0)} z_2}{4 r_2^5} \end{vmatrix},$$

which reduces to

$$J = (\Delta_2^{(0)})^8 \left[\Delta_2^{(0)} + (\lambda_2 x_2 + \mu_2 y_2 + \nu_2 z_2) \frac{3K_2^{(0)}}{4 r_2^5} \right].$$

It follows from the second, fifth, and eighth equations of (16) that

$$\lambda_2 x_2 + \mu_2 y_2 + \nu_2 z_2 = \rho_2 + \lambda_2 X_2 + \mu_2 Y_2 + \nu_2 Z_2$$
$$= \rho_2 - R_2 \cos \psi .$$

Therefore

$$(58) \qquad J = (\Delta_2^{(0)})^8 \left[\Delta_2^{(0)} + (\rho_2 - R_2 \cos \psi) \frac{3 K_2^{(0)}}{4 r_2^5} \right].$$

If either of the two factors of J is zero the solution as a power series in τ in general fails. If the first factor is zero (the case in which the apparent motion of the observed body is in the arc of a great circle) the equations for ρ_1 and ρ_3 no longer involve these quantities. If the second factor is zero the solution for $\tau = 0$ is multiple. Since ρ_1, ρ_3, x_2, y_2, z_2, x_2', y_2', and z_2' are uniquely determined by equations (43), (48), and (50) after r_2 has been computed, the multiplicity all lies in the determination of r_2. The value of r_2 is determined from the first of (44) and the geometrical condition that the Sun, observer, and observed body, shall form a triangle at $t = t_2$; or

$$\begin{cases} \Delta_2^{(0)} \rho_2 - K^{(0)} - \dfrac{K_2^{(0)}}{4 r_2^3} = 0, \\ r_2^2 - R_2^2 - \rho_2^2 + 2R_2 \rho_2 \cos \psi = 0. \end{cases}$$

The Jacobian of these functions with respect to ρ_2 and r_2 is $2r_2$ times the second factor of (58). Perhaps this is the easiest method of approach to the condition for multiple solutions. However, the discussion which has been given furnishes directly all the conditions under which the form of the solution changes.

In *Meddelande från Lunds Observatorium*, No. 46, CHARLIER has given the "singular surfaces" on which the Lagrangian solution fails. It corresponds to the second factor of (58)*. The condition was arrived at, however, through explicit development of the series.

Suppose the second factor of (58) is zero. The solutions will then in general be developable as power series in $\sqrt{\tau}$. There will be two exact solutions of the equations both of which will be either real or complex. If the data belong to a physical problem there will be one real solution, and consequently in this case there will be also a second.

While a power series solution of the problem is thus shown to be possible, it is not advisable to compute orbits in this way because of the long developments in series which would be required. The analytic solution has no theoretical advantages over the method of successive approximations, for they are simply alternative methods of arriving at the solution of simultaneous equations, and they are subject to the same limitations.

17. FORMULAS FOR COMPUTATION.

In the actual computation of an orbit only a small part of the formulas are used which have appeared in the course of the many theoretical and practical investigations which are contained in this paper. On this account it is in order to gather together those which are actually to be used.

* The Lagrangian method also fails if $\Delta_2^{(0)} = 0$.

And a few remarks are due on the uncertainties to which the final results are subject, and to the modifications they should introduce into the computations.

The Data. It is supposed that at the epochs t_1, t_2, t_3, which are not separated from one another by long intervals, the coördinates α_i and δ_i ($i = 1,2,3$), are given by observation, and that they have been corrected for precession, aberration, etc. It is also supposed that the corresponding X_i, Y_i, and Z_i have been taken from the *Nautical Almanac* and corrected for the position of the observer on the Earth. Then λ_i, μ_i, and ν_i are computed from

$$(6)\qquad \begin{cases} \lambda_i = \cos\delta_i \cos\alpha_i\,, & (i = 1,2,3). \\ \mu_i = \cos\delta_i \sin\alpha_i\,, \\ \nu_i = \sin\delta_i\,. \end{cases}$$

If the α_i and δ_i are given correctly by the observations only to seconds of arc, as will be the case when the observed body is a comet, six-place tables should be used; but if the α_i and δ_i are correct to tenths of seconds of arc, seven-place tables should be used.

The Determinants. The following determinants are required:

$$(20)\qquad \Delta_2 = \begin{vmatrix} \lambda_1+\lambda_3-2\lambda_2\,, & \lambda_2\,, & \lambda_3-\lambda_2 \\ \mu_1+\mu_3-2\mu_2\,, & \mu_2\,, & \mu_3-\mu_2 \\ \nu_1+\nu_3-2\nu_2\,, & \nu_2\,, & \nu_3-\nu_2 \end{vmatrix},$$

$$(39)\qquad \begin{cases} K = \begin{vmatrix} \lambda_1\,, & X_1+X_3-2X_2\,, & \lambda_3-\lambda_1 \\ \mu_1\,, & Y_1+Y_3-2Y_2\,, & \mu_3-\mu_1 \\ \nu_1\,, & Z_1+Z_3-2Z_2\,, & \nu_3-\nu_1 \end{vmatrix}, \\[2ex] K_1 = \begin{vmatrix} \lambda_1\,, & X_1-X_3\,, & \lambda_3-\lambda_1 \\ \mu_1\,, & Y_1-Y_3\,, & \mu_3-\mu_1 \\ \nu_1\,, & Z_1-Z_3\,, & \nu_3-\nu_1 \end{vmatrix}, \\[2ex] K_2 = \begin{vmatrix} \lambda_1\,, & X_1+X_3\,, & \lambda_3-\lambda_1 \\ \mu_1\,, & Y_1+Y_3\,, & \mu_3-\mu_1 \\ \nu_1\,, & Z_1+Z_3\,, & \nu_3-\nu_1 \end{vmatrix}, \end{cases}$$

$$(43)\qquad \begin{cases} K_3 = -\begin{vmatrix} X_1+X_3-2X_2\,, & \lambda_2\,, & \lambda_3-\lambda_2 \\ Y_1+Y_3-2Y_2\,, & \mu_2\,, & \mu_3-\mu_2 \\ Z_1+Z_3-2Z_2\,, & \nu_2\,, & \nu_3-\nu_2 \end{vmatrix}, \\[2ex] K_4 = \begin{vmatrix} X_3-X_2\,, & \lambda_2\,, & \lambda_3-\lambda_2 \\ Y_3-Y_2\,, & \mu_2\,, & \mu_3-\mu_2 \\ Z_3-Z_2\,, & \nu_2\,, & \nu_3-\nu_2 \end{vmatrix}, \\[2ex] K_5 = \begin{vmatrix} X_2\,, & \lambda_2\,, & \lambda_3-\lambda_2 \\ Y_2\,, & \mu_2\,, & \mu_3-\mu_2 \\ Z_2\,, & \nu_2\,, & \nu_3-\nu_2 \end{vmatrix}, \\[2ex] K_6 = \begin{vmatrix} X_1+X_3-2X_2\,, & \lambda_1\,, & \lambda_2-\lambda_1 \\ Y_1+Y_3-2Y_2\,, & \mu_1\,, & \mu_2-\mu_1 \\ Z_1+Z_3-2Z_2\,, & \nu_1\,, & \nu_2-\nu_1 \end{vmatrix}, \\[2ex] K_7 = \begin{vmatrix} X_2-X_1\,, & \lambda_1\,, & \lambda_2-\lambda_1 \\ Y_2-Y_1\,, & \mu_1\,, & \mu_2-\mu_1 \\ Z_2-Z_1\,, & \nu_1\,, & \nu_2-\nu_1 \end{vmatrix}, \\[2ex] K_8 = \begin{vmatrix} X_2\,, & \lambda_1\,, & \lambda_2-\lambda_1 \\ Y_2\,, & \mu_1\,, & \mu_2-\mu_1 \\ Z_2\,, & \nu_1\,, & \nu_2-\nu_1 \end{vmatrix}. \end{cases}$$

The number of places to be used in the calculations is determined before these determinants are computed by the number of significant figures in the smallest column of Δ_2. Sometimes other linear combinations of the column and rows than that suggested will still further reduce the number of significant figures in a column. It will rarely be necessary to use more than five-place tables in computing a preliminary orbit based on observations not separated by considerable intervals, and usually four-place tables will suffice.

It follows from the form of the determinants that it is advantageous to compute them in the three groups, viz., Δ_2, K_3, K_4, K_5; K, K_1, K_2; K_6, K_7, K_8. Since the minors of the members of each group which depend only on the λ_i, μ_i, and ν_i are respectively identical the computation is quickly and easily made.

Determination of r_2 and ρ_2. The values of r_2 and ρ_2 are determined in the first approximation from

$$R_2\cos\psi = -\lambda_2 X_2 - \mu_2 Y_2 - \nu_2 Z_2\,, \qquad (45)$$

$$\left.\begin{aligned} N\sin m &= R_2\sin\psi\,, \\ N\cos m &= R_2\cos\psi - \frac{1}{2\Delta_2}\left(K + \frac{\epsilon}{\tau}K_1\right), \end{aligned}\right\} \qquad (46)$$

$$\Phi = \frac{\tau^2}{4}P\,K_2 + \frac{\tau\epsilon}{12}Q\,K_1\,. \quad (P = Q = 1 \text{ in 1st approx.}) \qquad (44)$$

$$M = \frac{N\,\Delta_2\,R_2^3\sin^3\psi}{\Phi}\,, \quad (\text{sign of } N \text{ such that } M > 0)\,, \qquad (46)$$

$$\sin^4\varphi = M\sin(\varphi + m)\,, \qquad (47)$$

$$\left.\begin{aligned} r_2 &= R_2\frac{\sin\psi}{\sin\varphi}\,, \qquad (r_2^3 u = 1)\,, \\ \rho_2 &= R_2\frac{\sin(\psi+\varphi)}{\sin\varphi}\,. \end{aligned}\right\} \qquad (45)$$

Computation of ρ_1 and ρ_3. These quantities are determined from

$$
(42)\quad \left\{
\begin{aligned}
F_1 &= \frac{2\dfrac{\epsilon^2}{\tau^2} - u\tau\left(1 - \dfrac{\epsilon}{\tau}\right)(\tau P + \tfrac{1}{3}\epsilon Q)}{1 + \dfrac{\epsilon}{\tau} + \tfrac{1}{2}u\tau(\tau P + \tfrac{1}{3}\epsilon Q)}\,, \\
F_3 &= \frac{2\dfrac{\epsilon^2}{\tau^2} - u\tau\left(1 + \dfrac{\epsilon}{\tau}\right)(\tau P - \tfrac{1}{3}\epsilon Q)}{1 - \dfrac{\epsilon}{\tau} + \tfrac{1}{2}u\tau(\tau P - \tfrac{1}{3}\epsilon Q)}\,, \\
G_1 &= \frac{-2\dfrac{\epsilon^2}{\tau^2} + \tfrac{1}{3}u\tau\epsilon Q - u\epsilon(\tau P + \tfrac{1}{3}\epsilon Q)}{1 + \dfrac{\epsilon}{\tau} + \tfrac{1}{2}u\tau(\tau P + \tfrac{1}{3}\epsilon Q)}\,, \\
G_3 &= \frac{-2\dfrac{\epsilon^2}{\tau^2} - \tfrac{1}{3}u\tau\epsilon Q + u\epsilon(\tau P - \tfrac{1}{2}\epsilon Q)}{1 - \dfrac{\epsilon}{\tau} + \tfrac{1}{2}u\tau(\tau P - \tfrac{1}{3}\epsilon Q)}\,,
\end{aligned}
\right.
$$

where in the first approximation $P = Q = 1$. Then

$$
(43)\quad \left\{
\begin{aligned}
\Delta_2\,\rho_1 &= K_3 + 2\frac{\epsilon}{\tau}K_4 + F_1K_5 + G_1(K_4 + K_5)\,, \\
\Delta_2\,\rho_3 &= K_6 + 2\frac{\epsilon}{\tau}K_7 + F_3K_8 + G_3(K_8 - K_7)\,.
\end{aligned}
\right.
$$

Reduction of the time. The corrected values of t_1, t_2, t_3 are

$$t_1 - 498^s.65\rho_1\,, \qquad t_2 - 498^s.65\,\rho_2\,, \qquad t_3 - 498^s.65\,\rho_3\,.$$

If only five-place tables are used in the computation it might be supposed that this correction would be unnecessary because only the differences of the epochs of the observations are involved, except in the determination of the time of perihelion passage. The correction should be made provided $\dfrac{500\,(\rho_3 - \rho_1)}{t_2}$ is equal to or greater than one unit in the last significant figure retained; an inspection is sufficient to decide the matter.

The question of the reduction of the time is closely associated with that of deciding how great accuracy in determining the instant of an observation is required. This is obviously a question of importance for the observer. The errors arising from errors in the determination of the times of the observations enter in two ways. In the first place they obviously enter in the determination of τ and ϵ. In order that the data may all be of the same order of exactness these quantities must be determinate to the same number of significant figures as the smallest column in Δ_2. For intervals of six days, when τ is of the order of 0.1, the second order column in Δ_2 is generally of the order of 0.01 in the case of comets, and of 0.001 in the case of small planets. Consequently, if the observations are very accurate, giving the apparent places of the respective bodies to seconds and tenths of seconds of arc, then, in order to be correspondingly exact, the times of the observations must be given at least to seconds of time.

The second way in which errors in determining the times of the observation affect the results is through the X_i, Y_i, and Z_i. It may be supposed that these quantities can be obtained from the Ephemeris with any desired degree of precision for the dates in question. Hence they are subject only to the errors coming from the errors in determining t_1, t_2, and t_3. These errors are important in K, K_3, and K_6 because these determinants have second order columns which are functions of the coördinates of the Earth alone. It follows from equations (37), (39), and (43) that the errors introduced in this way are as important as those entering directly through τ and ϵ.

While the determinants are under consideration it may be noted that the shorter the intervals between the observations the more important are the effects upon the elements of errors both of apparent position and also of the times of the observations. The relationship is that the possible errors in the determination of the elements vary, because of the presence of second order columns in the determinants, inversely as the squares of the intervals between the observations. Now since perturbations of position of the observed body are, for short intervals of time, proportional to the second power of the time, it follows that their effects upon the elements of an orbit, as determined from three observations, are as important for short intervals as for longer ones.

Computation of x_2, y_2, z_2, x_2', y_2', z_2'. These quantities are determined by

$$
\left.
\begin{aligned}
x_2 &= \lambda_2\,\rho_2 + X_2\,, \\
y_2 &= \mu_2\,\rho_2 + Y_2\,, \\
z_2 &= \nu_2\,\rho_2 + Z_2\,;
\end{aligned}
\right\} \quad (48)
$$

$$
\left.
\begin{aligned}
(g_3 - g_1)\,x_2' &= X_3 - X_1 - (f_3 - f_1)\,x_2 \\
&\qquad + \lambda_1(\rho_3 - \rho_1) + (\lambda_3 - \lambda_1)\,\rho_3\,, \\
(g_3 - g_1)\,y_2' &= Y_3 - Y_1 - (f_3 - f_1)\,y_2 \\
&\qquad + \mu_1(\rho_3 - \rho_1) + (\mu_3 - \mu_1)\,\rho_3\,, \\
(g_3 - g_1)\,z_2' &= Z_3 - Z_1 - (f_3 - f_1)\,z_2 \\
&\qquad + \nu_1(\rho_3 - \rho_1) + (\nu_3 - \nu_1)\,\rho_3\,;
\end{aligned}
\right\} \quad (49)
$$

$$
\left.
\begin{aligned}
f_3 - f_1 &= 2\tau\,[-u\,\epsilon + \tfrac{1}{2}u\,p\,\tau^2 + \tfrac{3}{2}u\,p\,\epsilon^2 \\
&\qquad + \tfrac{1}{6}u(u - 15p^2 + 3q)\,\tau^2\epsilon \\
&\qquad - \tfrac{1}{8}up\,(u - 7p^2 + 3q)\,\tau^4\ \ldots]\,, \\
g_3 - g_1 &= 2\tau\,[1 - \tfrac{1}{6}u\,\tau^2 - \tfrac{1}{2}u\,\epsilon^2 + up\tau^2\epsilon \\
&\qquad + \tfrac{1}{120}u\,(u - 45p^2 + 9q)\,\tau^4\ \ldots]\,,
\end{aligned}
\right\} \quad (50)
$$

in which p and q are unknown in the first approximation. If it is desired to represent the observations exactly the number of places to which the data are given should be used in computing these quantities.

The second approximation. In order to make the second approximation p, q, P, and Q are computed from

$$(14)\qquad \begin{cases} r_2^2\, p = x_2 x_2' + y_2 y_2' + z_2 z_2' \ , \\ r_2^2\, q = x_2'^2 + y_2'^2 + z_2'^2 - \dfrac{1}{r_2} \ ; \end{cases}$$

(36)

$$\begin{cases} P = 1 - \dfrac{\epsilon^2}{\tau^2} - 2p\epsilon + \frac{1}{12}(7u - 15p^2 + 3q)\,\tau^2 \ldots . \\ Q = 1 - \dfrac{\epsilon^2}{\tau^2} + \frac{3}{2}p\dfrac{\tau^2}{\epsilon} - 3p\epsilon + \frac{1}{60}(37u - 765p^2 + 153q)\,\tau^2 \\ \qquad + \frac{1}{4}p\,(3u + 14p^2 - 6q)\dfrac{\tau^4}{\epsilon} \quad \ldots\ldots \end{cases}$$

With these quantities determined the computation proceeds exactly as in the first approximation. The approximation may be continued until equations (16) are satisfied when the elements, which are to be determined from x_2, y_2, . . . , z_2' by the usual process, will represent the observations. Ordinarily the second approximation will be sufficient.

18. Numerical Illustrations.

However clear and certain a proposition may be mathematically, astronomers like to see numerical illustrations of it. For this reason, and to give a check on the signs and numerical coefficients of the formulas, some practical problems will be briefly considered.

The second of the three orbits quoted in section 1 was based upon the following data:

t	α	δ
May 7.8757	312° 39′ 37″.5	+ 10° 57′ 5″
May 9.8633	310 51 43 .5	12 47 29
May 11.8470	308 48 30 .0	14 47 7

It is found by equations (6) that

$$\begin{array}{lll} \lambda_1 = +0.665311 & \mu_1 = -0.721991 & \nu_1 = +0.189976 \\ \lambda_2 = +0.638004 & \mu_2 = -0.737518 & \nu_2 = +0.221402 \\ \lambda_3 = +0.605966 & \mu_3 = -0.753445 & \nu_3 = +0.255197 \end{array}$$

Then the expression for Δ_2 becomes

$$\Delta_2 = \begin{vmatrix} \lambda_1 + \lambda_3 - 2\lambda_2\,, & \lambda_2\,, & \lambda_3 - \lambda_2 \\ \mu_1 + \mu_3 - 2\mu_2\,, & \mu_2\,, & \mu_3 - \mu_2 \\ \nu_1 + \nu_3 - 2\nu_2\,, & \nu_2\,, & \nu_3 - \nu_2 \end{vmatrix}$$

$$= \begin{vmatrix} -.004731\,, & +.638004\,, & -.032038 \\ -.000400\,, & -.737518\,, & -.015927 \\ +.002369\,, & +.221402\,, & +.033795 \end{vmatrix}$$

Since the first column contains only four significant figures the value of Δ_2 is determined only to four significant figures at best. In fact, it is determined only to three as is seen by adding twice the third row of the determinant to the first, and six times the second row to the third. Therefore the elements are determined by these data only to three significant figures, corresponding to about 0°.1 in the angular elements when they are not subject to additional partial indetermination because of small inclination or eccentricity. These conclusions are fully illustrated by comparing the last two sets of elements given in section 1. The computation of the elements used in the data for the second set could have been made by the formulas of this paper with four-place tables without any loss of real accuracy.

A second example, which has been considered is that given in Watson's *Theoretical Astronomy*, pp. 264-277. As a starting point the data on p. 266 were adopted, the ecliptic being the fundamental plane of reference. While the elements are really determinate only to five places at the most, it was decided in order to test the accuracy of the formulas to assume that the observational and ephemeris data are absolutely correct and to make the computation with seven places. The data of the problem, as given by Watson, are

t (1863)	Longitude	Latitude
257.68079	17° 46′ 28″.17	+3° 8′ 43″.51
264.42570	16 40 25 .19	2 52 27 .62
271.38625	15 15 44 .03	2 32 42 .98

Long. of Sun	
172° 0′ 32″.23	$\log R_1 = 0.0021056$
178 35 48 .74	$\log R_2 = 0.0011656$
185 25 36 .90	$\log R_3 = 0.0002378$

From these data it is found that

$$\begin{array}{lll} \lambda_1 = +.9508307 & \mu_1 = +.3048115 & \nu_1 = +.0548703 \\ \lambda_2 = +.9567491 & \mu_2 = +.2865593 & \nu_2 = +.0501456 \\ \lambda_3 = +.9637793 & \mu_3 = +.2629775 & \nu_3 = +.0444088 \end{array}$$

$\Delta_2 = -.00000677049$.

$$\begin{array}{ll} X_1 = +\ .9951027 & Y_1 = -.1396940 \\ X_2 = +1.0023866 & Y_2 = -.0245526 \\ X_3 = +\ .9960627 & Y_3 = +.0946278 \end{array}$$

In the first approximation Watson found

$$\begin{array}{ll} \log r_2 = 0.3025672\ , & \log \rho_1 = 0.0254823\ , \\ \log \rho_2 = 0.0123991\ , & \log \rho_3 = 0.0028859\ . \end{array}$$

The final values of these quantities as determined by Watson are

$$\begin{array}{ll} \log r_2 = 0.3032587\ , & \log \rho_1 = 0.0269143\ , \\ \log \rho_2 = 0.0137621\ , & \log \rho_3 = 0.0041748\ . \end{array}$$

The first approximation by the method of this paper gave

$$\log r_2 = 0.3029767\,, \qquad \log \rho_1 = 0.0269235\,,$$
$$\log \rho_2 = 0.0137533\,, \qquad \log \rho_3 = 0.0040231\,.$$

Then it was found that

$$\log x_2 = 0.2988355\,, \qquad \log x_2' = 9.1894528n\,,$$
$$\log y_2 = 9.4333324\,, \qquad \log y_2' = 9.8727382\,,$$
$$\log z_2 = 8.7139865\,, \qquad \log z_2' = 8.7577158n\,.$$
$$\log u = 9.0910699\,,$$
$$\log p = 8.7122982n\,,$$
$$\log q = 8.3382518\,.$$

Since these results did not exactly satisfy equations (16) a second approximation was necessary, as was expected. In making the second approximation a numerical error was committed which was not noticed until it was completed. Hence a third approximation was necessary, which gave (the correction of time aberration was omitted since the only purpose was to test the formulas)

$$\log r_2 = 0.3030078\,,\ \log x_2 = 0.2988658\,,\ \log x_2' = 9.2059031n$$
$$\log \rho_1 = 0.0270538\,,\ \log y_2 = 9.4333994\,,\ \log y_2' = 9.8717483$$
$$\log \rho_2 = 0.0138146\,,\ \log z_2 = 8.7140478\,,\ \log z_2' = 8.7599206n$$
$$\log \rho_3 = 0.0041348\,,\ \log f_1 = 9.9996407\,,\ \log g_1 = 9.0644381n$$
$$\log u = 9.0909766\,,\ \log f_3 = 9.9996146\,,\ \log g_3 = 9.0780963$$

When these quantities are substituted in equations (16) five of them are exactly satisfied to the seventh decimal inclusive, three have an error of one unit in the seventh place, and one has an error of two units in the seventh place. Therefore the elements derived from them by the usual methods will represent all three observations to a corresponding degree of accuracy.

In the preceding computation it was virtually assumed that in the λ_i, μ_i and ν_i the digits beyond the seven places which are written are all zero. As a matter of fact they may have any values, not exceeding five units in the eighth place, which satisfy the conditions that $\lambda_1^2 + \mu_1^2 + \nu_1^2 = 1$, $\lambda_2^2 + \mu_2^2 + \nu_2^2 = 1$, $\lambda_3^2 + \mu_3^2 + \nu_3^2 = 1$. These possible variations in the λ_i, μ_i, and ν_i would not be important if Δ_2 were not small. In order to exhibit the actual order of its magnitude let it be written in the form

$$\Delta_2 = \begin{vmatrix} \lambda_1 + \frac{4}{5}\lambda_3 - \frac{2}{5}\lambda_2\,, & \lambda_2\,, & \lambda_3 - \lambda_2 \\ \mu_1 + \frac{4}{5}\mu_3 - \frac{9}{5}\mu_2\,, & \mu_2\,, & \mu_3 - \mu^2 \\ \nu_1 + \frac{4}{5}\nu_3 - \frac{9}{5}\nu_2\,, & \nu_2\,, & \nu_3 - \nu_2 \end{vmatrix}$$

$$= \begin{vmatrix} -.0002943\,, & \lambda_2\,, & +.0070302 \\ -.0006132\,, & \mu_2\,, & -.0235815 \\ +.0001352\,, & \nu_2\,, & -.0057368 \end{vmatrix}.$$

It follows that Δ_2 is determinate only to four significant figures and that the whole computation can be made with five-place tables without any real loss of accuracy. Since Δ_2 is determinate only to four places the elements are determinate only to four places. The angular elements were given by Watson to hundredths of seconds of arc, corresponding to the eighth decimal place. Therefore, the actual uncertainty in the elements is of the order of 5000 times one unit in the last place given by Watson.

Computation shows that the co-factors of each of the elements of the first column of Δ_2 are negative, and Δ_2 itself is negative. Consequently to diminish Δ_2 numerically as much as possible by changing the λ_i, μ_i, and ν_i four units in the eighth decimal it is necessary to diminish λ_1, λ_3, μ_2, ν_1, and ν_3, and to increase λ_2, μ_1, μ_3, and ν_2. Since the μ_i and ν_i are small compared to the λ_i the conditions $\lambda_1^2 + \mu_1^2 + \nu_1^2 = 1$, . . . will be satisfied to seven places after these changes have been made. It follows from the explicit expression for Δ_2 that each term of its expansion will be altered in the fifth significant figure. It follows from the composition of the first column from λ_1, λ_2, λ_3, . . . that the change will be $4\left(\frac{5}{5} + \frac{4}{5} + \frac{9}{5}\right) = 14$ units, and the total change from the three terms of the expansion of Δ_2 with respect to the elements of the first column will be about three times this quantity, or four units in the fourth significant figure.

With the indicated changes in the eighth place the expression for Δ_2 becomes

$$\Delta_2 = \begin{vmatrix} -.00029438\,, & .95674914\,, & +.00703020 \\ -.00061310\,, & .28655926\,, & -.02358180 \\ +.00013512\,, & .05014564\,, & -.00573680 \end{vmatrix}.$$

The third column is unchanged and the changes in the second column can only affect the eighth significant figure; hence the minors used in the preceding computation may be used. Then it is found that

$$\Delta_2 = -.00000676651$$

which differs from the value previously found by four units in the fourth significant figure. If the λ_i, μ_i and ν_i had been altered in the opposite direction in the eighth decimal place the change in Δ_2 would have been equally in the opposite direction.

The determinants K, K_1, and K_2 will be altered by these changes only in the eighth significant figures, and such small changes may be neglected. Then it is found, on using the values of P and Q previously computed, that

$$\log r_2 = 0.3031397\,, \qquad \log \rho_2 = 0.0140731\,,$$

which differ one and two units respectively in the fourth place from those which were obtained above. If the computation were completed they would require some subsequent slight changes through alterations in P and Q, but the order of the possible variations is established in complete accord with the previsions of theory.

SUNSPOT OBSERVATIONS,

MADE AT BERWYN, PENN., WITH A $4\frac{1}{2}$-INCH REFRACTOR,

BY ALDEN W. QUIMBY.

1913	Time	New Grs.	Total Grs.	Total Spots	Fac. Grs.	Def.
July 1	7					fair
2	7					fair
3	7					fair
4	7					fair
5	7					fair
6	7					fair
7	7					fair
8	7					fair
9	7	1	1	3		fair
10	4		1	4		fair
11	7		1	6		fair
12	7		1	6		fair
13	7					fair
14	7					fair
15	4				1	good
16	7				1	good
17	7					fair
18	7					fair
19	7					fair
20	7					fair
21	7					fair
22	7					fair
23	7					fair
24	7					fair
25	7					fair
26	7					fair
27	7					fair
28	7					fair
29	7					fair
30	7					fair
31	7					fair
Aug. 1	9					fair
2	7					fair
3	7					fair
4	7					fair
5	7					fair
6	7					fair
7	7					fair
8	7					fair
9	7					fair
10	7					fair
11	7					fair
12	7					fair
13	5					poor
14	7					fair
15	7					fair
16	7					fair
17	7					fair
18	7					fair
19	7					fair
20	7					fair
21	7					fair
22	7					fair
23	2					fair
24	7					fair
25	7					fair
26	7					fair
27	7					fair

1913	Time	New Grs.	Total Grs.	Total Spots	Fac. Grs.	Def.
Aug. 28	7					poor
29	7					poor
30	5					poor
31	7					poor
Sept. 1	7					fair
2	7					fair
3	7					fair
4	7					fair
5	7					fair
7	4	1	1	3	1	fair
8	7		1	5	1	fair
9	7		1	2	1	fair
10	7				1	fair
11	7					fair
12	9					fair
13	7					fair
14	7					fair
15	7					fair
16	7					fair
17	7					fair
18	7					fair
20	2					poor
21	9					fair
22	7					fair
23	7					fair
24	7					fair
25	7					fair
26	7					fair
27	7					fair
28	7					fair
29	8					fair
30	7					fair
Oct. 1	3					poor
2	7					fair
3	7					fair
4	3					poor
5	7					fair
6	7	1	1	1	6	fair
9	11					poor
10	7					fair
11	1					poor
12	4					fair
13	7					fair
14	7					fair
15	3					fair
16	7					fair
17	7					fair
18	7					fair
20	12					fair
21	7					fair
22	7					fair
23	7					fair
25	4					poor
26	12	1	1	9		fair
27	8		1	5		fair
28	8		1	5		fair
29	8					fair
30	8					fair

1913	Time	New Grs.	Total Grs.	Total Spots	Fac. Grs.	Def.
Oct. 31	8					fair
Nov. 1	8					fair
2	8					fair
3	8					fair
4	8					fair
5	8					fair
6	8					fair
7	8					fair
8	8					fair
9	2					fair
10	8					fair
11	2					fair
12	8					fair
13	8					fair
14	8					fair
15	8					fair
17	8					fair
18	8					fair
19	8					fair
20	8					fair
21	8					fair
22	8					fair
23	8					fair
24	8					fair
25	8					fair
26	8					fair
29	8					fair
Dec. 1	8					fair
2	12					fair
3	8					fair
4	10					fair
5	8					fair
6	2					fair
8	8					fair
9	8	1	1	4		fair
10	8		1	4		fair
11	8		1	6		fair
12	8		1	6		fair
13	8		1	3	1	fair
14	8				2	fair
15	8				2	fair
16	8				1	fair
17	8					fair
18	8					fair
19	8					fair
20	8					fair
21	8					fair
22	8					fair
23	8					fair
24	8					fair
25	8					fair
26	8					fair
27	8					fair
28	8					fair
29	8					fair
30	8	1	1	4		fair
31	3		1	5		fair

OBSERVED MAXIMA AND MINIMA OF VARIABLE STARS—1912-1913,

By PAUL S. YENDELL.

TV Cassiopeæ.

During the autumn of 1912 I secured two minima of *TV Cassiopeæ*, as follows:

1912 Sept. 9, fourteen observations, from $7^h\ 25^m$ to $10^h\ 40^m$. Minimum indicated at $8^h\ 5^m$.

Oct. 8, nine observations, from $7^h\ 23^m$ to $10^h\ 20^m$. Minimum indicated at $8^h\ 21^m$.

RZ Cassiopeæ.

During October and November of 1912 I observed three minima of *RZ Cassiopeæ*, as follows:

1912 Oct. 18, twelve observations, from $6^h\ 20^m$ to $8^h\ 33^m$. Minimum indicated at $7^h\ 48^m$.

Nov. 18, twelve observations, from $6^h\ 7^m$ to $10^h\ 50^m$. Minimum indicated at $9^h\ 37^m$.

30, six observations, from $5^h\ 43^m$ to $9^h\ 20^m$. Minimum indicated at $8^h\ 20^m$.

Z Vulpeculæ.

During the past season I have observed two minima of *Z Vulpeculæ*, as follows:

1913 Sept. 24, thirteen observations, from $7^h\ 40^m$ to $11^h\ 20^m$. Minimum indicated at $10^h\ 20^m$.

29, thirteen observations, from $7^h\ 23^m$ to $10^h\ 2^m$. Minimum indicated at $8^h\ 16^m$.

XZ Cygni.

I have three fairly well indicated maxima of this *Antalgol* star, observed during the past season, as follows:

1913 Aug. 23, seven observations, from $8^h\ 3^m$ to $9^h\ 58^m$. Maximum indicated at $9^h\ 3^m$.

Sept. 5, eleven observations, from $7^h\ 47^m$ to $11^h\ 0^m$. Maximum indicated at $10^h\ 31^m$.

6, twelve observations, from $7^h\ 32^m$ to $9^h\ 36^m$ Maximum indicated at $8^h\ 34^m$.

At all of these three maxima the star's highest observed light was $8^M.8$.

V Vulpeculæ.

During the year 1913 I observed *V Vulpeculæ* fifty times, from June 11 to Dec. 19, with considerable interruption, however, from weather and absence. These observations indicate two maxima and two minima, as follows: a maximum of $8^M.5$ on 1913 July 7.7, followed by a minimum of $9^M.4$ on July 21.3; from Aug. 3 to Aug. 19, a gap occurs in the observations; a minimum of $9^M.4$ is indicated on Sept. 3.0, and a maximum of $8^M.2$ on Sept. 27.1.

Following the second maximum there appeared to a halt in the light-change, the star remaining at about $8^M.6$ from Oct. 21 to Nov. 21, during which time it was observed eleven times. A similar halt occured from Aug. 19 to 25, during which time five observations were made, all at $9^M.0$.

In no case have I seen the star fainter than $9^M.4$.

Ch. 7792 *SS Cygni.*

During the years 1912 and 1913 I have made the following observations of *SS Cygni*:

		Mag.			Mag.			Mag.
1912 Oct.	5.333	8.29	1913 July	11.447	8.83	1913 Nov.	5.333	11*
	6.318	8.07		13.374	8.16		6.375	9.55
	7.339	8.39		14.407	8.09		.382	9.55
	8.322	8.15		15.367	8.70		7.267	9.15
	10.318	8.11		16.324	8.74		.330	8.92
	14.340	7.99	Maximum July 14.0				8.257	8.94
	15.340	9.14					9.345	8.81
	18.332	10±*	Sept.	10.330	8.25		11.344	8.36
	20.344	11±*		14.318	8.30		14.315	8.03† ::
Maximum Oct. 10.33							17.323	9.6
							21.330	11*

* Eye-estimates. † Bright moonlight; difficult and uncertain.

All dates given in the above paper are stated in Boston Mean Time, = Greenwich Mean Time $-4^h\ 44^m.5$.

Dorchester, January 3, 1914.

COMET b, 1912 (*SCHAUMASSE–TUTTLE.*)

By F. E. SEAGRAVE.

Elements.

$$
\begin{aligned}
E &= \text{Dec. } 25.50 \ 1912. \quad \text{G.M.T.} \\
M &= 4^\circ \ 2' \ 46''.74 \\
\omega &= 206^\circ \ 46' \ 59''.06 \\
\pi &= 116^\circ \ 37' \ 44''.26 \\
\Omega &= 269^\circ \ 50' \ 45''.20 \\
i &= 55^\circ \ 12' \ 50''.93 \\
\text{Log } e &= 9.916143 \\
\text{Log } a &= 0.767404 \\
\text{Log } q &= 0.011903 \\
\mu &= 250''.553
\end{aligned}
$$

Constants.

$$
\begin{aligned}
x &= r(9.75626) \ \sin\,(359^\circ \ 43' \ 47''.53 + u) \\
y &= r(9.98871) \ \sin\,(250^\circ \ 18' \ 42''.04 + u) \\
z &= r(9.93020) \ \sin\,(332^\circ \ 8' \ 7''.05 + u)
\end{aligned}
$$

Final Check Formulæ.

$$
q = a(1 - e) = \frac{p}{(1 + e)} = (a \cos \varphi) \tan\,(45^\circ - \tfrac{1}{2}\varphi)
$$

	0.767404		0.273026	0.520215
	9.244499		0.261125	9.491688
Log q	= 0.011903	=	0.011901	= 0.011903

Normal Positions upon which the Elements are Based.

$^{12}_{5}t_0 = 339.7474196$	$\lambda = 211^\circ 39' \ 2''.97$	$\beta = -40^\circ 28' 35''.91$
$^{12}_{13}t_0' = 347.7033477$	$\lambda' = 220^\circ 57' 47''.55$	$\beta' = -42^\circ \ 2' 16''.45$
$^{12}_{21}t_0'' = 355.8230816$	$\lambda'' = 229^\circ 31' 13''.24$	$\beta'' = -42^\circ 59' 17''.01$

$\odot = 253^\circ 31' 18''.10$	$\log R \ 9.993496$	$\psi' = 55^\circ 41' 54''.40$
$\odot' = 261^\circ 36' 22''.77$	$\log R' \ 9.993074$	
$\odot'' = 269^\circ 52' 15''.35$	$\log R'' 9.992813$	

Heliocentric Coördinates, etc., from Fourth and Last Hypothesis.

$l = 149^\circ 14' 32''.30$	$\log r = 0.072360$	$u = 251^\circ 21' 11''.39$
$l' = 160^\circ 30' \ 5''.26$	$\log r' = 0.094392$	$u' = 258^\circ 40' 27''.37$
$l'' = 171^\circ 49' 11''.34$	$\log r'' = 0.117896$	$u'' = 265^\circ 24' \ 2''.71$

$b = -51^\circ \ 5' 38''.76$	$\log \rho = 0.032299$	$z' = 40^\circ 51' 30''.60$
$b' = -53^\circ 38' 20''.18$	$\log \rho' = 0.045333$	$\log p = 0.273026$
$b'' = -54^\circ 56' 57''.96$	$\log \rho'' = 0.061517$	

$v = 44^\circ 34' 12''.33$	$E = 14^\circ 29' 29''.70$	$M = 2^\circ 40' 17''.57$
$v' = 51^\circ 53' 28''.31$	$E' = 17^\circ 10' \ 3''.30$	$M' = 3^\circ 13' 31''.02$
$v'' = 58^\circ 37' \ 3''.65$	$E'' = 19^\circ 45' 31''.82$	$M'' = 3^\circ 47' 25''.49$

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THE DIRECT AND SCATTERED RADIATION OF THE SUN AND STARS.

BY C. G. ABBOT.*

The last time I saw Prof. KAPTEYN he asked me if I could tell him how large a proportion of the light we get from the stars is direct, and how much is scattered by the atmosphere. Strictly speaking it is almost all scattered, for the stars by virtue of their distances should appear practically as points, but in fact they do appear as areas far larger than their actual dimensions would warrant. But I conceive that Prof. KAPTEYN was not dealing in such niceties as this, but regarded all the light which would fall in a good star image as direct, and that which the same star furnishes indirectly over the whole sky as scattered.

Undoubtedly the complete answer to the question would require the treatment of each wave length by itself, and would require us to know the energy distribution of the spectra of all the stars and the distribution of their brightness over the sky as it would be outside our atmosphere. We should get a different result at each elevation above sea-level, and a hazy night would differ from a clear one.

As it was quite impossible to deal so thoroughly with the question, I limited myself. (*a*) To the average transparency prevailing at Mount Wilson on Sept. 22 and Sept. 24, 1913. This is somewhat less than the mean transparency of many years there, but decidedly greater than the Mount Wilson transparency of 1912, after the Alaskan volcanic dust became widespread,† and is indeed decidedly above the transparency of a hazy night there in usual years. (*b*) To stars of the same energy spectrum distribution as the Sun. (*c*) To the total radiation of such stars as they would be measured by the bolometer. (*d*) To a collection of such stars distributed uniformly over the sky, so that outside the atmosphere the sky would be everywhere equally rich in stars.

The problem was approached by means of solar observations. Measurements were made on the relative brightness of the Sun and sky, for different altitudes of the Sun, and for different altitudes and azimuths of the sky. The instrument used comprised a bolometer exposing an area about 0.8^{mm} by 0.8^{mm}. This was mounted equatorially at the lower end of a tube about 1 meter long and 12 centimeters in diameter, provided with numerous diaphragms having apertures about 10 centimeters in diameter. A quartz lens of 9.5 centimeters clear diameter, and 30 centimeters focus, projected the image of a portion of the Sun, or of the sky, upon the exposed area of the bolometer. The sky was generally allowed to shine with full intensity through the quartz lens. In order to reduce the brightness of the Sun to a similar intensity, one of two diaphragms, with apertures of 3.26 and 5.88 millimeters diameter, respectively, was placed centrally over the end of the tube, a rotating sector whose aperture was 0.0450 that of a complete circle was inserted in the beam, and a resistance of 100 ohms was put in series with the galvanometer, instead of 1 ohm usually employed when observing the sky. As the bolometer measured merely the central part of the Sun's image, which by other work I have found to be brighter than the average Sun in the ratio $\frac{406}{352}$, this factor was introduced. Altogether the Sun observations with the smaller aperture were multiplied by 0.000000304 to make them comparable with those on the sky.

The equatorial instrument was mounted on the tower of the Smithsonian observing station on Mount Wilson, and pointed and operated by my colleague, Mr. L. B. ALDRICH. The writer made the galvanometer readings and records, and reduced the observations.

On the afternoon of September 22, 1913, three series of pointings were made on the sky. Two consisted of 36 and 35 pointings to all parts of the sky, and the third comprised 18 pointings, taking every odd numbered pointing of the second series. The Sun was observed from time to time, and one special set of 10 pointings very near the Sun was made. On the forenoon of Sept. 24, practically the same program was repeated, but with the special set of pointings close up to the Sun twice carried through.

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† It appears that the transparency above Mount Wilson in 1913 was somewhat less than that of the years preceding the outbreak of Mount Katmai.

On the whole the observations are very consistent, and as the two days gave nearly the same indications for similar positions in the sky, the results have been combined.

In the reductions, the sky positions were reduced to altitude and azimuth, counting the momentary position of the Sun as of zero azimuth. Ratios were taken of the observations of brightness, so as to express the brightness of the sky at each position as a fraction of the momentary mean brightness of the Sun, for equal angular areas. The results were then collected in groups with respect to altitude. In illustration I give the following values obtained in the first series of September 22.

Table I. Brightness of Sky in Zone of Altitude 10° to 20°.

Altitude	Azimuth (from Sun)	Ratio $\left(\frac{\text{Sky}}{\text{mean Sun}}\right)$
14° 37′	50° 00′	356 x 10^{-8}
14° 37′	34° 30′	465
17° 54′	92° 40′	200
17° 54′	− 4° 15′	660
16° 20′	164° 10′	204
15° 41′	−156° 10′	214
14° 30′	− 76° 40′	248

Sky : Mean altitude 15° 56′.
Sun : Mean altitude 46° 0′.

Six such groups of sky observations were arranged, covering the following altitude ranges: 0°−5°; 5°−10°; 10°−20°; 20°−30°; 30°−45°; 45°−65°. There was besides a single observation at altitude 85° 42′.

Each such group of observations was plotted with brightness ratios as ordinates and azimuths (from Sun) as abscissæ. Symmetrical curves with respect to the azimuth zero were thus defined. By measuring the area included under each such curve the mean brightness ratio for each altitude zone in question was obtained. Thus were found the following results to represent the six series of sky observations of Sept. 22 and Sept. 24. It should be remarked that for the two short series, only five groups were employed.

The figures show some discordances, but it was difficult to draw the curves of azimuth and brightness, in some cases, for lack of observations at critical points. On the whole the comparable results of the two days (by which I mean those of nearly equal solar hour angles) do not differ by above 14 per cent. on the average. The ratios are generally somewhat larger for the second day than for the first.

Each of the series, as given in Table II, was plotted with mean altitudes as abscissæ and mean ratios of brightness as ordinates. Smoothed curves were drawn, and these no doubt aided to improve the results somewhat. From these curves were taken off the values at sky altitudes 5°, 15°, 25°, 35°, 47½°, 65° and 82½° corresponding to each series. A difference between the results of the several series depending on the altitude of the Sun was very apparent. Accordingly for each of the above 7 altitudes a plot was made with values of the ratio $\left(\text{Mean } \frac{\text{sky}}{\text{sun}}\right)$ as abscissæ, and solar altitudes as ordinates. The solar altitudes ranged from 14½° to 46°. The trend of the results was well marked in this interval, and permitted the plots to be extrapolated with much confidence from solar altitude 0° to 90°. This being done, the following values were taken as representing the mean ratio of brightness, $\frac{\text{sky}}{\text{sun}}$ for various sky zones and for certain altitudes of the Sun.

Table II. Mean Brightness of Sky Zone.

September 22, 1913						September 24, 1913					
Series I		Series II		Series III		Series I		Series II		Series III	
Hour Angle of Sun 1h42m—2h28m		3h30m—4h7m		4h24m—5h1m		4h57m—4h38m		3h50m—3h16m		2h12m—1h40m	
Mean Altitude	Mean Sky/Sun	Mean Altitude	Mean Sky/Sun	Mean Altitude	Mean Sky/Sun	Mean Altitude	Mean Sky/Sun	Mean Altitude	Mean Sky/Sun	Mean Altitude	Mean Sky/Sun
2° 47′	550 × 10^{-8}	2° 47′	482 × 10^{-8}	4° 36′	642 × 10^{-8}	4° 36′	766 × 10^{-8}	2° 47′	779 × 10^{-8}	2° 47′	601 × 10^{-8}
7° 18′	366	7° 18′	515	. . .	. . .	. . .	. . .	7° 18′	642	7° 18′	506
15° 56′	333	15° 56′	421	15° 50′	378	15° 50′	465	15° 56′	451	15° 56′	364
26° 5′	263	26° 5′	312	29° 46′	264	29° 46′	346	26° 5′	326	26° 5′	317
36° 46′	253	34° 36′	294	44° 59′	161	44° 59′	166	36° 36′	247	36° 36′	263
57° 15′	256	55° 12′	228	64° 16′	122	64° 16′	116	56° 29′	192	56° 29′	243
85° 42′	176	85° 42′	114		. . . . .	. . .	. . .	85° 42′	114	85° 42′	192

Table III. Mean Ratio of Brightness $\left(\frac{\text{Sky}}{\text{Sun}}\right) \times 10^8$.

Sun's Altitude	Sky Altitude in Zones. 0–10°	10°–20°	20°–30°	30°–40°	40°–55°	55°–75°	75°–90°
5°	**1200** / 750	490	375	250	130	84	52
15°	690	**840** / 460	355	253	160	122	92
25°	630	432	**652** / 340	258	200	157	130
35°	573	404	320	**511** / 263	230	190	168
47½°	500	370	298	265	**399** / 258	236	218
65°	390	320	260	275	300	**113** / 300	280
82½°	295	270	230	282	340	350	**736** / 358
Area of Zone in Hemispheres	0.174	0.168	0.158	0.143	0.176	0.147	0.034

In Table III a line of bold faced figures is seen running diagonally across. The ordinary figures in the same boxes represent the results as thus far explained. Those in bold face type are introduced to allow for the extraordinary brightness of the sky near the Sun. They were obtained in the following manner. As stated above, three sets of observations were made close to the Sun. By means of these observations it was found that the immediate neighborhood of the Sun was brighter than was allowed for in the general reductions of the sky measurements. This extra brightness was a function of the distance from the Sun. Accordingly the region within 11° of the Sun was divided into a number of rings whose areas in terms of a hemisphere were known. Using these data, a weighted mean value of the extra brightness near the Sun was determined. In illustration of this I give the following values from the first determination of Sept. 24. The Sun's altitude was then 24° 10′.

Table IV. Brightness of the Sky Near the Sun.

Solar distance	1°.5	2°	3°	4°	5°	6°	8°	10°
Ratio $\frac{\text{Sky}}{\text{Sun}} \times 10^8$	23500	16300	8700	5200	3700	2900	2200	1700
Line 2 minus 1000*	22500	15300	7700	4200	2700	1900	1200	700
Ring limits	0°–1°.5	1°.5–2°.5	2°.5–3°.5	3°.5–4°.5	4°.5–5°.5	5°.5–7°	7°–8°	9°–11°
Area of ring $\times 10^5$ in hemispheres	33	61	92	121	152	285	486	606
Line 3 times line 5 $\times 10^8$	9.9**	9.3	7.1	5.1	4.1	5.4	5.8	4.2

Sum of line 5 = 184 $\times 10^{-4}$ hemispheres.

Sum of line 6 = 50.9 $\times 10^{-8}$

Mean brightness ratio for .0184 hemispheres:

$$\frac{50.9 \times 10^{-8}}{184 \times 10^{-4}} = 2770 \times 10^{-8}.$$

* This 1000 is the mean brightness ratio already estimated for this region in the general sky work, so that the figures in line three give what is over and above close to the Sun.

** Estimated brightness for this ring 30,000 $\times 10^{-8}$.

This and the other two series of observations near the Sun gave the following data:

Average extra brightness within 11° of Sun, $\frac{\text{sky}}{\text{sun}}$	1410×10^{-8}	2770×10^{-8}	3140×10^{-8}
Altitude of Sun	42° 20′	24° 10′	19° 0′

Additional observations at higher Sun would have been desirable, but, lacking these, the following values were estimated, after plotting the values given above:

Sun altitude	5°	15°	25°	35°	47½°	65°	82½°
Average value of extra brightness for 11° : $\left(\frac{\text{sky}}{\text{sun}}\right) \times 10^8$	1300	3500	2650	1900	1350	900	700

If a zone of the sky extending from 10° to 20° of altitude be considered to include the Sun at an altitude of 15°, a circle 11° in radius about the Sun would only partially be included in the zone mentioned. But for simplicity, and in view of the limitations of accuracy of the investigation, I have assumed that it is only the zone 10° to 20° in altitude which is made brighter by the extra brightness near the Sun when the Sun is at 15° altitude. Thus I have increased the number representing the mean brightness of each zone by finding the weighted mean of its brightness, as given by the general sky observations, and its added brightness as furnished by the special observations near the Sun, giving weights in proportion to the areas of the zone and of the circle of 11° radius, respectively. Thus, in illustration, for the zone 10° to 20° I have:

$$\frac{3500 \times .0184 + 460 \times .168}{.168} = 840$$

This number, 840 is found in bold faced type in Table III above.

In what has been given above the quantities of radiation furnished by angular areas of the sky equal to that of the Sun are compared with the quantities of radiation furnished in the direct solar beam at particular altitudes of the Sun. I now propose to express the intensity of the direct solar beam at different altitudes of the Sun in terms of the solar constant of radiation.

All observations of the Sun's brightness made during the two days Sept. 22 and Sept. 24, 1913, were collected in connection with the secants of the Sun's zenith distance at the moments of observation. A plot was made with values of secant Z as abscissæ, and logarithm solar intensity as ordinates. The tangent of the inclination of the best straight line representing the points was read off. Thus was obtained the apparent transmission coefficient of the atmosphere, for the Sun's total radiation, and for the days in question.* The value obtained in this way was 0.867. Tables 33 to 36 of Volume III, *Annals of the Smithsonian Astrophysical Observatory*, were then searched for similar values. On Sept. 18, 1909, the pyrheliometry yielded the value 0.865. On this day the solar constant value, by spectro-bolometric investigation, was 1.921 calories per sq. cm. per minute. The following pyrheliometer readings were made:

Secant Z	2.765	2.267	1.894	1.628	1.421	1.244	1.238
Calories observed	1.160	1.257	1.335	1.375	1.430	1.422	1.438

So far as they go these values undoubtedly represent closely the values which would have been obtained if the pyrheliometer had been read on September 22 and September 24, 1913, at equal altitudes of the Sun. Unfortunately these values do not reach to very low Sun. In order to get probable pyrheliometer values at very low Sun, a comparison was made between the values given above and those obtained on July 6, 1910, when the pyrheliometer was read from sunrise to noon.† Combining the results obtained by this comparison with those just given, the following values were chosen as the most probable intensities of solar radiation corresponding to different solar altitudes for September 22 and September 24, 1913.

* See *Annals*, Astrophysical Observatory, Vol. III, p. 99.

† See Abbot's "The Sun" (Appletons, 1911), p. 287.

Table V. Brightness of the Sun and Other Data.

Sun's altitude	5°	15°	25°	35°	47½°	65°	82½°
Sun's intensity in calories per cm² per min.	0.533	0.900	1.233	1.358	1.413	1.496	1.521
Sun's intensity if outside atmosphere it is 1.0	0.277	0.468	0.641	0.706	0.734	0.778	0.791
Limits of zones	0°–10°	10°–20°	20°–30°	30°–40°	40°–55°	55°–75°	75°–90°
Area of zones in hemispheres	0.174	0.168	0.158	0.143	0.176	0.147	0.034
Area of zones in square degrees	3590	3470	3260	2950	3630	3030	700
Product (line 3 × line 5)	0.0482	0.0786	0.1013	0.1010	0.1292	0.1144	0.0269
Quotient × 10^7 (line 7 : line 6)	134	227	311	342	356	378	384

The last five lines of Table V relate to Prof. KAPTEYN's problem of the stars, and will be explained a little later. Before taking up that problem, let us go a little further with the question of Sun and sky radiation by day.

By combining Tables III and V the total radiation of the sky and Sun for the whole hemisphere may be determined. If one wishes to consider the rays as falling on a horizontal surface, account has to be taken of the cosine of the zenith distance. Take, in illustration, the conditions when the Sun is at $47\frac{1}{2}°$ altitude:

TABLE VI. BRIGHTNESS OF DAY SKY FOR SOLAR ALTITUDE $47\frac{1}{2}°$.

Zone altitude	0–10°	10°–20°	20°–30°	30°–40°	40°–55°	55°–75°	75°–90°
Area in hemispheres	0.174	0.168	0.158	0.143	0.176	0.147	0.034
Cosine zenith distance	0.087	0.259	0.423	0.574	0.737	0.906	0.991
Ratio $\frac{\text{Sky}}{\text{Sun}} \times 10^8$	500	370	298	265	399	236	218
Product line 4 × line 2	87.0	62.2	47.1	37.9	70.2	34.7	7.4
Product line 5 × line 3	7.6	16.1	19.9	21.8	51.7	31.4	7.3

Taking the sum of line five from left to right we find that the mean brightness of the whole sky for equal areas is 346×10^{-8} times that of the Sun at $47\frac{1}{2}°$ altitude. As the Sun at this time occupied 108×10^{-7} hemispheres, and there are approximately 20630 square degrees in a hemisphere, and as the Sun (by Table V) furnished 1.413 calories per sq. cm. per minute, it follows that a square degree of sky furnished on the average

$$\frac{346 \times 10^{-8}}{108 \times 10^{-7}} \times \frac{1.413}{20630} = 219 \times 10^{-7} \text{ calories}$$

per sq. cm. per minute.

Taking the sum of line six from left to right we find that as received on a horizontal surface, the mean brightness of the whole sky for equal areas is 156×10^{-8} times that of the direct Sun rays at $47\frac{1}{2}°$ altitude, received at normal incidence. The whole sky therefore would furnish

$$\frac{1.413 \times 156 \times 10^{-8}}{108 \times 10^{-7}} = 0.205 \text{ calories per sq. cm. per minute}$$

on a horizontal surface. The Sun would furnish to such a surface $1.413 \times \text{cosine } 42\frac{1}{2}° = 1.041$ calories per sq. cm. per minute. The total from Sun and sky on a horizontal surface is 1.246 calories per sq. cm. per minute. Proceeding in a similar way with the other data of Tables III and V, we reach the following results:

TABLE VII. BRIGHTNESS OF SUN AND SKY.

Altitude of Sun	5°	15°	25°	35°	$47\frac{1}{2}°$	65°	$82\frac{1}{2}°$
Sun's brightness cal. per cm.² per min.	0.533	0.900	1.233	1.358	1.413	1.496	1.521
Sun's brightness on horizontal surface cal. per cm.² per min.	0.046	0.233	0.524	0.780	1.041	1.355	1.507
Mean brightness $\frac{\text{Sky}}{\text{Sun}} \times 10^8$. Normal.	423	403	385	365	346	326	310
Mean brightness of sky on horizontal surfaces $\frac{\text{Sky}}{\text{Sun}} \times 10^8$. Sun Normal.	115	132	142	150	156	163	170
Mean sky on normal. Calories per sq. degree × 10⁷, per cm.² per min.	101	163	213	222	219	219	211
Total sky on horizontal. Cal. per cm². per min.	0.056	0.110	0.162	0.189	0.205	0.226	0.240
Total Sun and sky on horizontal. Cal. per cm². per min.	0.102	0.343	0.686	0.969	1.246	1.581	1.747

We are now ready to take up Prof. KAPTEYN's problem. Imagine 1,000 equally bright stars of solar-type spectrum uniformly distributed over the hemisphere, and let their total brightness (each being observed on a surface normal to its beam) be one unit, of some arbitrary scale, if observed outside our atmosphere. The relative brightness of individual stars at different altitudes as observed on Mount Wilson, would be given by line three of Table V.

Taking into account the relative areas of the various zones named in Table V, as given in line five of the table, and the relative brightness of the individual stars as given in line three, we obtain the summation of the brightness of all the stars in the several zones, as given in line seven, Table V. By taking the sum of the numbers in this line we find that the direct radiation of all the stars in the hemisphere, as observed at Mount Wilson (each being observed on a surface normal to its beam), is 0.600, whereas if observed outside the atmosphere this would be 1.000. The last line of Table V gives the mean brightness of the direct light of the stars per square degree in the different zones, as they would be observed on Mount Wilson. Near the horizon the radiation is seen to be only one-third as bright as in the zenith, and, at the brightest, a square degree gives about 0.00004 as much radiation as the whole hemisphere outside our atmosphere. The mean brightness of the direct radiation of the stars per square degree for the whole hemisphere, at Mount Wilson is 291×10^{-7}, and outside the atmosphere it is 485×10^{-7}, in the arbitrary units above chosen.

We come now to consider the scattered radiation of the stars. Referring, in illustration, to Table VI, if the Sun were at $47\frac{1}{2}°$ altitude, and covering 108×10^{-7} hemispheres, an equal area of the sky in the zone 0° — 10° of altitude would furnish on the average 500×10^{-8} times as much light as the Sun. As the whole zone of 0° to 10° altitude occupies 0.174 hemispheres, its total scattered radiation, summed up on a normal surface, would amount to $\frac{0.174 \times 500 \times 10^{-8}}{108 \times 10^{-7}} = .0806$ times that of the Sun at $47\frac{1}{2}°$ elevation. But now we may substitute for the Sun at altitude $47\frac{1}{2}°$ the zone of stars lying between 40° and 55° of altitude, whose radiation, according to line seven of Table V, is 0.1292 times the radiation of all the stars we are considering outside the atmosphere. Accordingly, the zone of 0° to 10° altitude would contain scattered radiation, coming originally from the stars lying between 40° and 55°, amounting in all to $0.0806 \times 0.1292 = 0.0104$ times the total radiation outside the atmosphere of all the stars in the hemisphere. In a similar way we may find the contributions of scattered radiation from all the other zones of stars to illuminate the zone of 0° to 10° altitude, and taking their sum we obtain the total scattered star radiation in that zone.

I have performed this operation for all zones, and have reached the following results:

Table VIII. Direct and Scattered Star Radiation.

	Sky altitude	0–10°	10°–20°	20°–30°	30°–40°	40°–55°	55°–75°	75°–90°
	Total scattered star radiation in zone. Summed on normal surface.	0.0566	0.0422	0.0319	0.0242	0.0257	0.0184	0.0037
	Number of square degrees.	3590	3470	3260	2950	3630	3030	700
$\times 10^{-7}$	Scattered star radiation per square degree	158	122	98	82	71	67	53
$\times 10^{-7}$	Direct star radiation per square degree.	134	227	311	342	356	378	384
$\times 10^{-7}$	Total star radiation per square degree	292	349	409	424	427	445	437

If we take the sum of the numbers in line two of Table VIII, we find the total scattered star radiation of the whole hemisphere, as seen from Mount Wilson, and summed up everywhere on a surface normal to the line of sight. This is 0.2027 times the total radiation of the stars outside the atmosphere, similarly summed up. Hence the average scattered radiation per square degree is 98×10^{-7}. This is the same that prevails in the zone of 20° to 30° altitude. Scattered star radiation exceeds the direct in intensity for the zone 0° to 10° in altitude, but is only one-seventh as intense as the direct at the zenith. The total star radiation, direct and scattered combined, is almost of equal intensity all the way from 20° to 90° of elevation; but is only about 0.7 as intense as at these high altitudes in the zone 0° — 10°. Observations by Yntema and others have shown that in fact the sky at night is brighter near the horizon than at high altitudes. This apparent contradiction of my computations is not surprising, for it has been concluded by Yntema and others that the sky is illuminated largely at night by some terrestrial source, as well as by the stars.

It will doubtless be regretted by some that the figures given in this paper relate to the total radiation, and not to visible or photographic rays alone. I believe, however, that the results obtained will not differ much from what would have been found for homogeneous light at wave length 5,000 Angströms.

Summary.

In this paper I have given the results of bolometric measurements of the total radiation of the Sun and the

sky by day as observed on Mount Wilson, California, altitude 1,730 meters, on September 22 and 24th, 1913.

Table I gives an illustration of a group of measurements. Table II gives a preliminary reduction of the measurements made at different parts of the sky. In Table III, the average radiation of zones of the sky of different altitudes is compared with the radiation of the Sun, for different altitudes. In this Table, two sets of figures are given for the brightness of the zone which includes the Sun, one set uncorrected for the extreme brightness immediately surrounding the Sun, the other set corrected for this. Table IV contains a series of measurements of the sky very near the Sun, showing the relative quantities of radiation received from the sky and the Sun when the latter is at an altitude of 24°. The ratios give the brightness of the sky as actually observed, and also the excessive brightness which it was found to have over that which had been assumed had no measurements been made very close up to the Sun itself. Table V gives the intensity of the Sun's radiation in calories per square centimeter per minute for different altitudes above the horizon; also the ratio of the Sun's radiation at these altitudes to what would have been found outside the atmosphere altogether. The values are further reduced for use in the latter part of the paper. In Table VI is collected the information relating to the brightness of the sky in different zones as compared with the Sun when the Sun's altitude is $47\frac{1}{2}$° above the horizon. The table is given in illustration of the method of weighting the brightness ratios in connection with the area of the sky which they may be taken to represent. In Table VII a summary is given of the average and total brightness of the sky in terms of the brightness of the Sun for different altitudes of the Sun from the horizon to the zenith.

In the latter part of the paper the data which has been collected in regard to the relative brightness of the Sun and the sky is used to compute the proportion of light from the stars which is received directly by the observer to that which is received indirectly by scattering from the atmosphere. This information is collected together in Table VIII, which gives the scattered star radiation per square degree; the direct star radiation per square degree; and the total star radiation per square degree, for all zones of the sky from the horizon to the zenith. In these applications of the work to the determination of star radiation, it is assumed that the stars are equally distributed over a hemisphere; that they are all of the solar type of spectrum; and that the transmission of the atmosphere is comparable to that which occurred on Mount Wilson on September 22 and 24, 1913. The measurements relate also to the *total radiation* of stars of the solar type, and not to the photographic or the visual rays. However, it is believed that the results would apply closely to yellow-green visible rays.

OBSERVATIONS OF COMETS,

MADE WITH THE 26-INCH EQUATORIAL OF THE U. S. NAVAL OBSERVATORY,

[Communicated by Captain J. L. JAYNE, U. S. Navy, Superintendent].

Date	Wash. M.T.	*	Comp.	Δα	Δδ	App. α	App. δ	log p Δ α	log p Δ δ	Red. to App. Pl. α	Red. to App. Pl. δ
				Comet 1913 *a* (Schaumasse)							
1913	h m s			m s	′ ″	h m s	° ′ ″			s	″
May 8	14 7 29	1	30, 6	+1 20.97	+ 1 56.0	20 47 24.76	+11 47 17.0	9.574*n*	0.658	+1.05	−11.7*
24	13 31 5	2	30, 6	+1 6.53	− 1 51.6	19 0 45.09	+31 31 17.4	9.267*n*	0.121	+1.99	−16.3*
28	13 20 16	3	30, 6	−1 8.20	− 4 30.5	18 11 11.83	+36 39 47.0	8.813*n*	9.536	+2.32	−15.9*
June 2	11 21 41	4	30, 6	+1 9.80	−11 45.7	16 59 13.31	+40 46 1.3	9.144*n*	9.281*n*	+2.72	−12.7*
9	12 23 33	5	30, 6	−2 32.10	+ 6 51.4	15 20 56.00	+41 0 34.0	9.526	9.605	+2.85	− 7.0*
29	9 29 10	6	30, 6	+0 51.38	+ 5 29.4	13 10 28.02	+30 19 50.4	9.551	0.339	+2.03	− 1.2*
				Comet 1913 *b* (Metcalf)							
Sept. 4	13 0 13	7	25, 5	−4 20.58	+ 5 23.3	6 46 36.75	+58 5 55.0	9.926*n*	0.623	+2.66	− 3.3*
9	15 17 46	8	30, 10	−1 12.58	− 0 56.3	6 35 41.28	+61 19 22.8	9.920*n*	9.483*n*	+3.19	− 4.4*
Oct. 4	13 6 18	9	18, 6	+0 50.02	− 3 48.1	22 27 7.56	−66 17 40.1	9.959	0.193*n*	+2.68	+24.0*
28	8 28 28	10	30, 6	−0 45.53	− 5 46.9	20 47 44.64	+ 6 54 1.2	9.387	0.678	+2.54	+13.4*
Nov. 2	8 39 17	11	30, 6	+1 38.47	− 1 50.7	20 45 53.45	+ 0 54 51.9	9.469	0.733	+2.60	+11.1*
				Comet 1913 *c* (Neujmin)							
Sept. 9	13 15 27	12	30, 10	+0 19.33	− 2 11.8	23 48 1.55	+ 0 59 21.6	8.922	0 731	+3.48	+22.1†
22	11 23 3	13	30, 10	+1 7.95	− 1 29.6	23 39 51.90	+ 6 18 39.8	8.349*n*	0.673	+3.51	+22.5†
23	11 16 34	14	30, 10	−1 8.93	− 2 36.4	23 39 17.40	+ 6 40 18.6	8.420*n*	0.669	+3.52	+22.7†
24	11 28 47	15	30, 10	−0 30.36	− 2 52.4	23 38 42.60	+ 7 1 45.9	7.865	0.664	+3.52	+22.8†
Oct. 22	10 29 0	16	30, 10	+0 53.47	+ 2 49.4	23 33 40.83	+14 1 33.0	9.080	0.572	+3.41	+25.3*

Date	Wash. M.T.	*	Comp.	$\Delta\alpha$	$\Delta\delta$	App. α	App. δ	log $p\Delta$ α	log $p\Delta$ δ	Red. to App. Pl. α	Red. to App. Pl. δ
				Comet 1913 *d* (Westphal-Delavan)							
Sept. 27	12 9 6	17	30, 6	+2 58.75	+ 3 46.6	21 50 37.65	− 1 37 32.6	9.477	0.752	+3.40	+15.5†
Oct. 4	9 48 58	18	30, 10	+0 33.22	+ 6 36.4	21 29 24.92	+ 4 17 30.7	9.153	0.698	+3.14	+15.7*
				Comet 1913 *e* (Zinner)							
Oct. 26	7 59 28	19	24, 8	+0 38.19	+ 3 26.6	18 56 28.12	− 7 26 49.8	9.551	0.779	+2.27	− 0.6*
Nov. 1	8 6 50	20	30, 6	+1 8.61	− 4 4.2	19 27 13.67	−13 13 55.1	9.560	0.803	+2.48	− 0.1*
4	7 27 52	21	25, 5	−2 28.57	− 8 17.2	19 44 6.48	−16 13 42.2	9.482	0.830	+2.64	+ 0.4*
6	7 28 42	22	30, 6	+2 1.31	− 4 4.5	19 56 6.25	−18 15 47.1	9.479	0.839	+2.70	+ 0.2*

Observers: * C. B. Watts. † H. E. Burton.

Mean Places of Comparison Stars for 1913.0.

*	α	δ	Authority	*	α	δ	Authority
	h m s	° ′ ″			h m s	° ′ ″	
1	20 46 2.74	+11 45 32.7	A.G. Leipzig I 8167	12	23 47 38.74	+ 1 1 11.3	A.G. Nicolajew 5900
2	18 59 36.57	+31 33 25.3	A.G. Leiden 7058	13	23 38 40.44	+ 6 19 46.9	A.G. Leipzig II 11744
3	18 12 17.71	+36 44 33.4	A.G. Lund 7581	14	23 40 22.51	+ 6 42 32.3	A.G. Leipzig II 11756
4	16 58 0.79	+40 57 59.7	A.G. Bonn 10889	15	23 39 9.44	+ 7 4 15.5	A.G. Leipzig II 11746
5	15 23 25.25	+40 53 49.6	A.G. Bonn 9957	16	23 32 43.95	+13 58 18.3	A.G. Leipzig I 9376
6	13 9 34.61	+30 14 22.2	A.G. Leiden 4853	17	21 47 35.50	− 1 41 34.7	A.G. Nicolajew 5523
7	6 50 54.67	+58 0 35.0	A.G. Helsingfors-Gotha 4813	18	21 28 48.56	+ 4 10 38.6	A.G. Albany 7535
8	6 36 50.67	+61 20 23.5	A.G. Helsingfors-Gotha 4674	19	18 55 47.66	− 7 30 15.8	A.G. Wien-Ottakring 6499
9	22 26 14.86	+66 21 1.2	A.G. Christiania 3598	20	19 26 2.55	−13 9 50.8	A.G. Cambridge, U. S. 6830
10	20 48 27.63	+ 6 59 34.7	A.G. Leipzig II 10426	21	19 46 32.41	−16 5 25.4	A.G. Washington 7462
11	20 44 12.38	+ 0 56 31.5	A.G. Nicolajew 5277	22	19 54 2.24	−18 11 42.8	B.D. −18°5553 compared with A.G. Washington 7496. $\Delta\alpha = +2^m 15^s.99$ $\Delta\delta = -13' 13''.4$

NOTES.

1913 *a*

May 8. No nucleus visible. Comet rather faint. Hazy.
May 24. Nucleus visible. Comet faintly visible in 2-inch finder. Estimated magnitude between 9 and 10. Seeing fair.
May 28. Nucleus very plain. Seeing fair.
June 2. Bright nucleus. Seeing good.
June 9. Bright nucleus. Seeing fair.
June 29. Seeing good.

1913 *b*

Sept. 1. Very faint nucleus.
Sept. 9. Seeing poor.
Oct. 4. Seeing poor.
Oct. 28. Comet very faint at times. Clouds. Seeing fair.
Nov. 2. Comet rather faint and diffused. Seeing fair.

1913 *c*

Sept. 9. Bright nucleus. Faint nebulosity following. Seeing fair
Sept. 22. Bright nucleus. Faint nebulosity. Seeing fair.
Sept. 23. Bright nucleus. Faint nebulosity. Seeing excellent.
Sept. 24. Bright nucleus. Faint nebulosity. Seeing good.
Oct. 22. Comet very faint. Nothing but nucleus visible. Seeing fair.

1913 *d*

Sept. 27. Bright nucleus. Visible in 2-inch finder.
Oct. 4. Seeing poor.

1913 *e*

Oct. 26. Very diffused. Short tail. Seeing poor. Clouds.
Nov. 1. Seeing poor.
Nov. 4. Very faint and diffused. Moonlight. Seeing poor.
Nov. 6. Rather faint. Nucleus invisible. Moonlight. Seeing fair.

CONTENTS.

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OBSERVATIONS OF COMET 1913 *c* (NEUJMIN)

By E. E. BARNARD.

It is safe to say that a week or two after its actual discovery by photography, this object would not have been found by the regular comet seeker, for the absence of the feeble nebulosity, which at first formed a faint tail to the nucleus, would have made it impossible to distinguish the comet from the ordinary star. In the first observations with the 40-inch telescope it consisted of a star-like nucleus of the $11\frac{1}{2}$ magnitude, from which a feeble nebulosity extended for a couple of minutes to the south-east, thus forming a faint tail. There was no condensation of the nebulous matter about or towards the nucleus, which seemed to lie in the preceding edge of the comet. Indeed the nebulosity, though an actual part of the comet, did not seem to have any other than accidental connection with the nucleus (*A. N.*, 4690). This nebulosity soon became faint and disappeared entirely, leaving only the nucleus, which was perfectly stellar. That the nebulosity was fairly noticeable at the first is evident from the fact that when guiding with the 5-inch telescope of the Bruce photographic instrument on September 10 and 26 the comet looked like a small, faint, hazy spot, in whose preceding side the nucleus was occasionally visible. It was mainly on the nebulosity that the guiding was done. On the nights of September 8 and 9 the nebulosity was noticeable with the 12-inch telescope. The two photographs made with the Bruce telescope show the nebulosity more or less distinctly. The first, on September 10, was given an exposure of $1^h\ 31^m$ with a poor sky, and shows the tail very feebly. The other, on September 26, a fine night (when the nebulosity was much fainter), with $4^h\ 20^m$ exposure, shows the nucleus and tail about as I had seen them with the 40-inch telescope.

On two dates (September 22 and October 31), photographs of the region of the comet were made with the Bruce telescope. The guiding was done on a star. The comet left a strong trail, which, on each date, was perfectly sharp and free from nebulosity and would, unquestionably, have been taken for that of an asteroid. At the observation of the comet on September 23 the nucleus was compared in brightness with a small star near it. The two were equal. But the trail of the comet on the plate of September 22 was much stronger at any point than the image of this same star. The nucleus must, therefore, have been very much brighter, photographically, than the star.

Because of its unique appearance and of the fact that it proved to be periodic, I have secured every possible position of this comet. The star-like appearance of the nucleus, and the absence of nebulosity after the first few observations, made it impossible to identify the object by its appearance alone. The very accurate ephemerides of Messrs. EINARSSON and NICHOLSON of the Students' Observatory, Berkeley, California (which were printed in *Lick Observatory Bulletins*, Nos. 236 and 245), have, therefore, been invaluable in the observations.

Positions of the Comet.

1913	G.M.T.	$\Delta\alpha\cos\delta$	$\Delta\alpha$	$\Delta\delta$	Comps.	α Appt.	δ Appt.	Red. to Appt.		*
	h m s	″	m s	′ ″		h m s	° ′ ″	s	″	
Sept. 9	18 48 41	121.12	+0 8.08	−1 25.7	5 6	23 48 0.87	+ 0 59 45.9	+3.47	+22.1	1
9	20 14 25	83.58	+0 5.57	+0 10.9	5 7	23 47 58.36	+ 1 1 22.5	+3.47	+22.1	1
13	19 15 6	94.15	−0 6.28	−0 56.6	5 - 6	23 45 27.28	+ 2 46 1.0	+3.49	+22.2	2
14	19 18 56	. .	−0 37.57	+1 50.3	12 tr. – 6	23 44 49.41	+ 2 11 41.2	+3.49	+22.2	3
23	19 48 43	161.46	−0 10.84	−3 44.9	5 8	23 39 11.63	+ 6 43 18.8	+3.51	+22.7	1
25	17 40 18	. . . .	. . . .	−1 10.1	4	. . . .	+ 7 23 25.9	. . .	+22.9	5

1913	G.M.T.	$\Delta\alpha \cos\delta$	$\Delta\alpha$	$\Delta\delta$	Comps.	α Appt.	δ Appt.	Red. to Appt.		*
	h m s	"	m s	′ ″		h m s	° ′ ″	s	″	
Sept. 25	17 50 0		+1 51	..	2 tr.	23 38 8	...	+3.51	....	5
25	18 9 46	238.36	+0 16.02	−1 27.1	8– 7	23 38 7.57	+ 7 23 50.4	+3.51	+22.9	6
25	18 26 0		..	−1 13.7	. – 1	... ..	+ 7 24 3.8	....	+22.9	6
27	17 15 32	100.53	−0 6.77	−1 9.1	6– 6	23 37 6.08	+ 8 3 12.7	+3.51	+23.1	7
30	15 28 9	94.17	−0 6.36	−1 43.4	5 6	23 35 44.51	+ 8 58 31.7	+3.50	+23.1	8
Oct. 5	16 57 19	. .	−2 3.81	+2 29.1	12 tr. 6	23 33 54.74	+10 24 57.0	+3.50	+23.7	9
7	18 35 26	286.01	−0 19.41	+3 13.6	6 8	23 33 22.91	+10 57 5.4	+3.48	+23.9	10
7	18 49 13	288.41	−0 19.58	.	3	23 33 22.74	. . ..	+3.48	.. .	10
11	16 14 8	54.83	−0 3.74	−1 8.3	5–10	23 32 45.95	+11 52 56.5	+3.46	+24.2	11
12	15 45 13	94.37	−0 6.43	−0 45.3	5 6	23 32 40.90	+12 6 3.3	+3.46	+24.3	12
12	17 24 19	101.84	−0 6.94	.	1 .	23 32 40.39	.. ... .	+3.46	....	12
14	15 57 51	22.28	−0 1.52	+0 52.4	5 10	23 32 38.2	+12 31.8	+3.45	+24.4	13
14	16 7 45			+0 56.8	... 3	. ....	+12 31.8	....	+24.4	13
Nov. 1	16 9 58	173.56	+0 12.01	−2 21.7	5– 8	23 37 52.97	+15 31 15.3	+3.35	+26.2	14
1	17 22 8	.	+0 55.48	+1 20.2	8 tr.– 7	23 37 54.51	+15 31 39.8	+3.35	+26.2	15
2	15 32 26	51.63	−0 3.57	+2 49.2	6– 8	23 38 27.19	+15 38 59.2	+3.35	+26.3	16
4	14 5 49	160.70	+0 11.14	+0 39.4	5 8	23 39 39.91	+15 53 56.5	+3.34	+26.5	17
8	17 24 9	250.68	+0 17.12	+0 42.6	11 6	23 42 35.97	+16 24 20.8	+3.32	+26.7	18
8	17 39 31		.. .	+0 45.8	.. 6	. .....	+16 24 24.0	..	+26.7	18
15	15 52 22	64.73	+0 4.52	−0 45.3	6– 8	23 48 28.30	+17 11 30.7	+3.30	+27.2	19
15	16 35 26	75.49	−0 5.27	−4 18.6	8– 5	23 48 29.89	+17 11 42.5	+3.30	+27.2	20
16	16 13 27	78.19	+0 5.46	+0 59.6	9 8	23 49 25.32	+17 18 6.0	+3.29	+27.3	21
22	16 28 37	104.04	−0 7.29	−0 14.8	10– 5	23 55 27.6	+17 58.1	+3.28	+27.8	22
23	16 48 6	65.89	+0 4.62	−0 40.3	5–10	23 56 32.8	+18 4.2	+3.27	+27.8	23
Dec. 16	12 32 20	143.21	−0 10.18	−0 21.9	7–10	0 25 12.97	+20 20 0.1	+3.26	+28.9	24
16	13 42 56	85.29	−0 6.06	−0 4.8	8– 5	0 25 17.09	+20 20 17.2	+3.26	+28.9	24
20	16 7 22	4.75	−0 0.34	−3 41.8	13– 7	0 31 5.29	+20 45 11.8	+3.26	+29.0	25
20	16 43 2		. ...	−3 42.7	... 3	. . . .	+20 45 10.9	....	+29.0	25
21	15 57 25	313.26	+0 22.34	+0 50.3	8 9	0 32 31.03	+20 51 14.5	+3.25	+28.9	26
30	12 20 25	14.56	−0 1.05	+0 47.4	10 6	0 45 36.64	+21 45 43.9	+3.27	+29.2	27
30	12 43 22	5.79	+0 0.42	.....	8 .	0 45 38.11	....	+3.27	.....	27

Mean Places of Comparison-Stars.

*	α 1913.0	δ 1913.0	Authority
	h m s	° ′ ″	
1	23 47 49.32	+ 1 0 49.5	Nicolajew A.G.C. 5901
2	23 45 30.07	+ 2 46 35.4	13 magnitude. Compared with Albany A.G.C. 8163
3	23 45 23.48	+ 3 9 28.6	B.D. + 2° 4722. Compared with Albany A.G.C. 8165
4	23 39 18.96	+ 6 46 41.0	B.D. + 6° 5191. Compared with Leipzig A.G.C. 11751
5	23 36 13.22	+ 7 24 13.1	Leipzig A.G.C. 11728
6	23 37 48.01	+ 7 24 54.6	10 magnitude. Compared with Leipzig A.G.C. 11728
7	23 37 9.34	+ 8 3 58.7	10 magnitude. Compared with Leipzig A.G.C. 11754
8	23 35 47.37	+ 8 59 52.0	10½–11 magnitude. Compared with Leipzig A.G.C. 11709
9	23 35 55.05	+10 22 4.2	Leipzig A.G.C. 9397
10	23 33 38.84	+10 53 27.9	Leipzig A.G.C. 9382
11	23 32 46.23	+11 53 40.6	10½ magnitude. Compared with Leipzig A.G.C. 9351
12	23 32 43.87	+12 6 24.3	10½ magnitude. Compared with Leipzig A.G.C. 9354
13	23 32 36.3	+12 30.5	12½ magnitude. Compared with B.D. +12° 5009
14	23 37 37.61	+15 33 10.8	12 magnitude. Compared with W.B. 23^h 701
15	23 36 55.68	+15 29 53.4	W.B. 23^h 701
16	23 38 27.41	+15 35 43.7	12 mgnitude. Compared with Star 14
17	23 39 25.43	+15 52 50.6	12½ magnitude. Compared with Berlin A.G.C. 9670
18	23 42 15.23	+16 23 11.5	11 magnitude. Compared with Berlin A.G.C. 9695
19	23 48 20.18	+17 11 48.8	12 magnitude. Compared with Berlin A.G.C. 9720
20	23 48 31.86	+17 15 33.9	Berlin A.G.C. 9720
21	23 49 16.57	+17 16 39.1	11 magnitude. Compared with Berlin A.G.C. 9720
22	23 55 31.6	+17 57.8	11 magnitude. Compared with B.D. +17° 5019

*	α 1913.0	δ 1913.0	Authority
	h m s	° ′ ″	
23	23 56 24.9	+18 4.4	12 magnitude. Compared with Star 22
24	0 25 19.89	+20 19 53.1	12½ magnitude. Compared with Berlin A.G.C. 139
25	0 31 2.37	+20 48 24.6	12 magnitude. Compared with Berlin A.G.C. 173
26	0 32 5.44	+20 49 55.3	10½ magnitude. Compared with Berlin A.G.C. 173
27	0 45 34.42	+21 44 27.3	13 magnitude. Compared with Berlin A.G.C. 248

Measured Positions of Comparison Stars.

(COMP. STAR — KNOWN STAR).

Comp. Star	Known Star	Δα	Δα cos. δ	Comps.	Δδ	Comps.
		m s	″		′ ″	
2	Albany 8163	+0 56.57	.	18 tr.	+4 45.4	7
3	Albany 8165	+0 37.67	. . .	.	+6 43.0	.
4	Leipzig 11751	−0 11.45	170.61	5	+0 37.5	4
6	Leipzig 11728	+1 34.82	.	14 tr.	+0 41.5	4
7	Leipzig 11754	−2 55.31		8 tr.	+2 38.2	4
8	Leipzig 11709	+3 2.74		10 tr.	−1 8.0	4
11	Leipzig 9351	+2 49.39		14 tr.	+0 27.7	9
12	Leipzig 9354	+2 24.82		12 tr.	+2 11.4	4
13	B.D. + 12° 5009	−1 30.03	.	10 tr.	−0 10.0	3
14	W. B. 23^h 701	+0 41.93		8 tr. 13	+3 17.4	4
16	Star 14	+0 49.80	.	16 tr.	+2 32.9	4
17	Berlin 9670	+1 6.97		16 tr.	+1 42.2	5
18	Berlin 9695	−0 51.98	.	16 tr.	−4 6.1	4
19	Berlin 9720	−0 11.38	163.08	6	−3 45.1	4
21	Berlin 9720	+0 44.71	.	24 tr.	+1 5.2	8
22	B. D. + 17° 5019	+0 17.30	247.11	. . .	+6 59.4	.
23	Star 22	+0 53.32	.	. . .	+6 40.5	..
24	Berlin 139	−1 37.69	.	20 tr.	−1 3.5	4
25	Berlin 173	−0 17.51	245.48	6	−2 36.5	6
26	Berlin 173	+0 45.56	.	18 tr.	−1 5.8	4
27	Berlin 248	+0 54.45	.	18 tr.	+5 11.3	4

In the above table, under the heads of "comps.," where the number of comparisons is omitted, a step star was used. The missing number of comparisons will be found with the measures of the step stars. In the case of star 14 a faint step star was used, and the Δα was partly transits and partly direct measures, while in the Δδ measures the step star was omitted.

Following are the measures of the intermediate or step stars which were used in some of the observations:

a − Albany 8165 (Nov. 4 $\Delta\alpha + 30^s.69$ (14 tr) $\Delta\delta + 2'\ 59''.9$ (4)
Dec. 16 $+2\ 59\ .9$ (5)
$+2\ 59\ .9$

a − star 3 (Nov. 4 $\Delta\alpha \cos\delta$ $94''.25$ (4) $= \Delta\alpha - 0^m 6^s.99$ $\Delta\delta$ $-3'\ 42''.7$ (4)
Dec. 16 $-3\ 43\ .4$ (4)
$-3\ 43\ .0$

The measures with Albany 8165 give for the place of a (= 12½ magnitude):

1913.0 α $23^h\ 45^m\ 16^s.49$ $\delta + 3°\ 5'\ 45''.5$

b − BD + 17° 5019 Nov. 22 $\Delta\alpha \cos\delta$ $265''.60$ (4) $= \Delta\alpha + 18^s.60$ $\Delta\delta + 5'\ 7''.8$ (6)

b − Star 22 Nov. 22 18 .49 (5) = $+\ 1\ .30$ $-1\ 51\ .6$ (5)

From the comparison with the BD star we get for the position of b (= 13.8 magnitude).

$$1913.0\ \alpha\ 23^h\ 55^m\ 32^s.9\quad \delta + 17°\ 55'.9$$

c − Star 22 Nov. 23 $\Delta\alpha + 38^s.02$ (18 tr) $\Delta\delta + 4'\ 9''.2$ (5)

c − Star 23 Nov. 23 $\Delta\alpha \cos\delta\ 218''.16$ (5) $= \Delta\alpha - 15^s.30$ $\Delta\delta - 2'\ 31''.3$ (5)

Using Star 22 we have for the position of c (=9−10 magnitude), 1913.0 α $23^h\ 56^m\ 9^s.6$ $\delta + 18°\ 1'.9$.

By inadvertance, a wrong star was at first observed for the comparison star of September 13. This involved several other stars, all of which were directly or indirectly compared with Albany *A. G. C.*, 8165. Following are the results:

Star 1 − Albany 8165	$\Delta\alpha = -0^m\ 27^s.32$	$\Delta\delta = -4'\ 31''.6$
2 − Albany 8165	−0 14 .40	−5 48 .6
3 − Albany 8165	+0 0 .21	−2 52 .8

These give the following positions:

Star 1 (BD+2°4717)	1913.0	$23^h\ 41^m\ 18^s.18$	+ 2°58′ 14″.0
2 (10(?) mag.)	1913.0	23 44 31 .10	+ 2 56 57 .0
3 (13½ mag.)	1913.0	23 44 46 .01	+ 2 59 52 .8

Estimated Magnitudes of the Nucleus of the Comet.

Date	Mag.	Remarks
Sept. 9	11½	Sky good, seeing poor.
13	10½	Seeing bad. Nearly full moon.
14	10½	Seeing poor. Full moon near.
25	11	Sky clear, seeing very bad.
27	11½	Seeing bad.
30	12	Sky thick.
Oct. 5	12	Sky broken with clouds.
7	12.1	Sky poor, seeing fair.
11	12.2	Sky broken with clouds.
12	12.2	Sky fair, seeing poor.
14	12½	Faint from bad seeing and white sky.
Nov. 2	13.4	Sky fair.
4	14	Sky good, seeing good.
8		Very faint. Nearly full moon.
15	13½	Very faint. Sky poor and moonlit.
16	13½	Very faint in clouds and haze. Seeing bad
22	14	Very faint.
23	14.3	Faint. Seeing very bad.
Dec. 16	15	Very faint and difficult. Sky poor.
20	15	Very faint. Seeing very bad.
21	15	Sky good, seeing good.
30	16	Very faint and difficult. Sky poor, seeing good.

The last estimate (16th magnitude) may be too faint, because of the poor condition of the sky.

A few notes on the visual appearance of the comet are given here for the value they may have at future returns of this object.

September 8. The comet was seen with the 12-inch telescope. A note says: "It was faint and partly mixed up with one of several small stars." On the night of September 9 the comet was looked at with the 12-inch before it was observed with the 40-inch, and it was again found to be "mixed up with a small star." Even when first observed with the large telescope, I waited for the comet to leave the star so that it could be better observed, only to find, after waiting, that the "star" was going with the nebulosity and that it was really the nucleus of the comet.

September 13 and 14. No nebulosity was visible on these dates because of the presence of the nearly full moon near the place. The bright moon on September 23 also obliterated the nebulosity.

On September 24 with the 12-inch telescope, on a moonless but not very clear sky, only a feeble nebulosity was seen with the nucleus, which was faint and star-like.

September 25. Very feeble and diffused nebulosity extended south following.

September 27. There were feeble traces of nebulosity south following from the nucleus, which was perfectly star-like.

September 30. When best seen only the feeblest traces of nebulosity were visible south following. Sky poor.

October 5. No trace of nebulosity. Sky poor.

October 7. No trace of nebulosity. Sky not very transparent.

In all the succeeding observations the comet was entirely devoid of any traces of nebulosity. The nucleus was perfectly star-like (as it had always been from the first observations) and the comet could only be identified by its motion. This stellar appearance was especially noticeable on November 4, when the observations were on a transparent sky with good seeing.

The almost continuous cloudy weather prevented any effort to observe the comet after December 30.

From *Lick Observatory Bulletin* No. 250, in a comparison of observations of the comet with the orbit computed by Messrs. Einarsson and Nicholson, the observation of November 4 is shown to be discordant in right ascension by some 5″. I am unable to account for this discordance. I have redetermined the position of the comparison star from a photograph and it agrees closely with the place given here. In the observations of the comet there were two sets of $\Delta\delta$ and one set of $\Delta\alpha$. The settings for the $\Delta\alpha$ were checked immediately after the measures of the $\Delta\alpha$. The discordance is not large enough for an error of one revolution of the micrometer screw = 9″.665.

Having had occasion in the early observations of this comet to refer to the Algiers Astrographic charts, I found

that there is a large discordance where charts Nos. 177 and 178 overlap. The error or difference between these two charts amounts to several minutes of arc in both right ascension and declination.

In the photographic copies of the BD charts, where all known errors are corrected up to 1898, a star of about the eighth magnitude is located in 1855.0 α 23h 42m 0s δ +2° 27′. This star is not in the BD catalogue at that place, nor is it in the sky.

On September 9 the star Nicolajew 5900 (9m.0) was compared with Nicolajew 5901 (8m.4). The observed $\Delta\alpha$ and $\Delta\delta$ are discordant when compared with the catalog places. The two stars were, therefore, brought up from the different catalogs where they could be found, with the following results.

		h m s	° ′ ″
W. B. 23h903	1913.0	23 47 39.12	+1 1 13.3
Munich 32833		23 47 38.94	+1 1 9.5
Kam 4807		23 47 38.79	+1 1 9.4
Nicolajew 5900		23 47 38.74	+1 1 11.3
Mean =		23 47 38.90	+1 1 10.9

		h m s	° ′ ″
W. B. 23h907	1913.0	23 47 50.03	+1 0 51.8
Munich 32834		23 47 49.31	+1 0 54.0
Kam 4808–9		23 47 49.53	+1 0 51.5
Nicolajew 5901		23 47 49.32	+1 0 49.5
Mean =		23 47 49.55	+1 0 51.7

The mean $\Delta\alpha$ and $\Delta\delta$ from these observations are:

$\Delta\alpha = 0^m\ 10^s.65 \qquad \Delta\delta = 0'\ 19''.2$

My own observations give

$\Delta\alpha = 0^m\ 10^s.50\ (8\ tr) \qquad \Delta\delta = 0'\ 19''.5 \qquad (4)$

During the observations the following two nebulæ were found and measured.

Date	No.	$\Delta\alpha$ (neb. — ✱)	Comp.	$\Delta\delta$ (neb. — ✱)	Comp.	Comp. star
		m s		′ ″		°
1913 Nov. 22	1	−0 3.96	2	+3 39.6	2	BD+17 5013
23	2	+0 5.33	2	+4 57.9	3	+17 5018

Mean Places, etc., of the Nebulæ.

	α 1855.0	δ 1855.0	Description
	h m s	° ′	
1	23 50 4.4	+17 30.8	R, 10″ diam., 14 mag.
2	23 52 5.0	+17 23.2	R, ½′ diam., 14 mag. gbM.

On November 4, when measuring the position of the comparison star for September 14, BD + 2° 4719 = Albany 8165 was found to be double. The following measures have been obtained.

		P.A.	Dist.	Mags.	
		°	″		
1913.844	Nov. 4	181.8	1.63	8½	12
.997	Dec. 30	180.5	1.61	...	13
1913.920		181.1	1.62	8½	12.5

The W.B. and B.D. stars given in this paper, and perhaps some of the other stars, should be put on the observing list of some meridian circle.

Yerkes Observatory, Williams Bay, Wis., 1914 February 27.

OBSERVATIONS OF COMET *1913 f* (DELAVAN),

MADE WITH THE 26-INCH EQUATORIAL OF THE U. S. NAVAL OBSERVATORY,

By H. E. BURTON, Assistant in the Observatory.

[Communicated by Captain J. L. Jayne, U.S. Navy, Superintendent.]

Date	Wash. M.T.	✱	Comp.	$\Delta\alpha$	$\Delta\delta$	App. α	App. δ	log $p\Delta$ α	log $p\Delta$ δ	Red. to App. Pl. α	Red. to App. Pl. δ
1913	h m s			m s	′ ″	h m s	° ′ ″			s	″
Dec. 18	8 35 47	1	25 , 5	−3 58.67	+3 15.2	3 2 26.93	−7 19 57.0	8.886*n*	0.800	+4.13	+19 1
19	7 10 56	2	25 , 5	+1 27.17	−2 47.7	3 1 37.21	−7 14 30.9	9.358*n*	0.793	+4 10	+19.2
22	7 42 29	3	25 , 5	+1 6.77	−6 28.3	2 59 1.69	−6 56 8.9	9.157*n*	0.795	+4.07	+18.9
26	8 59 36	4	10 , 2	+3 23.70	+6 17.5	2 55 45.66	−6 29 37.0	8.676	0.795	+4.04	+18.6
27	7 59 4	5	25 , 5	−0 55.77	+2 4.9	2 55 1.50	−6 23 2.8	8.811*n*	0 794	+4.06	+18.5
29	7 50 26	6	30 , 6	+1 5.68	−7 59.5	2 53 32.35	−6 8 54.3	8.801*n*	0.792	+4.04	+18.6
1914 Jan. 5	8 8 12	7	25 , 5	+4 44.78	+1 4.1	2 48 50.73	−5 15 35.4	8.568	0.785	+0.94	+ 3.2
11	7 32 51	8	30 , 6	+0 46.95	+8 49.5	2 45 32.00	−4 26 3.0	8.301	0.779	+0.90	+ 2.9
18	7 40 55	9	25 , 5	−4 34.14	−2 47.8	2 42 29.03	−3 23 50.1	8.985	0.770	+0.81	+ 2.9
21	7 29 38	10	30 , 6	−1 47.70	+0 21.6	2 41 27.34	−2 56 7.4	8.998	0.766	+0.76	+ 2.7

Mean Places of the Comparison-Stars for the beginning of the year.

*	α	δ	Authority	*	α	δ	Authority
	h m s	° ′ ″			h m s	° ′ ″	
1	3 6 21.47	−7 23 31.3	A.G. Wien-Ottakring 728	6	2 52 22.63	−6 1 13.4	A.G. Wien-Ottakring 664
2	3 0 5.94	−7 12 2.4	A.G. Wien-Ottakring 696	7	2 44 5.01	−5 16 42.7	A.G. Strassburg 675
3	2 57 50.85	−6 49 59.5	A.G. Wien-Ottakring 685	8	2 44 44.15	−4 34 55.4	A.G. Strassburg 678
4	2 52 17.92	−6 36 13.1	A.G. Wien-Ottakring 662	9	2 47 2.36	−3 21 5.2	A.G. Strassburg 684
5	2 55 53.21	−6 25 26.2	A.G. Wien-Ottakring 677	10	2 43 14.28	−2 56 31.7	A.G. Strassburg 671

NOTES.

Dec. 18. Comet visible in 5-inch finder. Bright nebulosity. Seeing good

Dec. 19 Comet visible in 5-inch finder: not visible in 2-inch finder. Fairly bright nucleus surrounded by nebulosity. Seems to show trace of tail following. Apparently not quite as bright as on Dec. 18.

Dec. 22 Comet fainter than on Dec. 18 and Dec. 19, perhaps on account of haze. Faint nucleus and nebulosity. No tail. Seeing fair.

Dec. 26. Comet visible in 5-inch finder. Fairly bright nucleus surrounded by nebulosity. Brighter than on Dec. 22. No tail. Seeing poor.

Dec. 27. Rather bright nucleus surrounded by nebulosity. Brighter apparently than on Dec. 22. and Dec 26. Seeing good.

Dec. 29. Comet visible in 5-inch finder. Fairly bright nucleus surrounded by nebulosity. Apparently fainter than on Dec. 27. Seeing good

Jan. 5. Could not see comet in 5-inch finder. Fairly bright nucleus surrounded by faint nebulosity. Moonlight. Seeing fair.

Jan. 11. Could not distinguish comet in 5-inch finder. Faint nucleus surrounded by nebulosity. Moonlight. Seeing good.

Jan. 18. Comet visible in 5-inch finder, not visible in 2-inch finder. Fairly bright nucleus surrounded by faint nebulosity. Seeing fair. Sky a trifle hazy.

Jan. 21. Comet visible in 5-inch finder. Fairly bright nucleus surrounded by nebulosity. Seems to show trace of tail following as on Dec. 19. Seeing poor.

The brightness of the comet appeared to remain practically the same throughout the series of observations. Comet not visible in the 2-inch finder at any time.

OBSERVATIONS OF COMET 1911 *b* (KIESS),

MADE WITH THE 12-INCH REFRACTOR OF THE ARGENTINE NATIONAL OBSERVATORY, CORDOBA,

BY E. CHAUDET.

Greenwich Mean Time	* No.	No. of Comp.	Δα	Δδ	App. α	App. δ	log. pΔ α	log. pΔ δ	Red. to App. Pl. α	Red. to App. Pl. δ
1911 h m s			m s	′ ″	h m s	° ′ ″			s	″
Aug. 22 18 54 7	1	5		+7 19.3	...	−52 4 34.2		0.082	...	+5.9
22 19 13 56	1	5	−0 30.98	...	21 28 55.72	...	9.841	...	+4.57	...
22 19 37 53	2	6	−0 31.58	...	21 28 23.67	...	9.843		+4.58	...
22 20 22 54	3	4	...	−0 48.7		−52 7 42.4	...	9.897n	...	+5.7
23 15 19 31	4	3	...	+2 33.4	...	−52 36 38.1	...	0.505	...	+2.6
23 15 38 39	4	6	−1 12.41	...	21 3 17.72	...	8.932	...	+4.61	...
23 18 51 5	6	5	−0 16.05	...	20 59 30.82	...	9.831	...	+4.61	...
23 19 7 15	6	5	...	+4 10.6	...	−52 39 47.2	...	9.462	...	+2.1
24 16 47 9	7	4	...	+1 45.3	...	−52 16 19.2	...	0.376	...	−1.0
24 17 5 18	7	10	−0 12.30	...	20 35 46.68	...	9.682		+4.56	...
24 17 43 6	8	11	+0 13.23	...	20 35 9.03	...	9.769	...	+4.56	...
24 18 10 58	8	9		+1 32.7	...	−52 16 1.2	...	9.956	...	−1.1
25 13 54 39	9	1		−0 11.4		−52 38 9.3	...	0.501	...	−3.4
25 14 5 2	9	3	−0 23.15	...	20 16 37.56	...	8.748n	...	+4.46	...
25 14 41 15	9	5	−0 55.18	...	20 16 5.53	...	8.884		+4.46	...
25 16 18 58	10	7	...	+1 19.1	...	−52 36 29.4	...	0.380	...	−3.6
25 16 58 17	10	5	−0 37.85		20 14 10.37	...	9.726		+4.46	...
25 18 31 23	11	16	+0 15.71	...	20 12 52.38	...	9.869		+4.45	...
25 19 1 23	11	5		+3 57.0	...	−52 34 25.9	...	9.943n	...	−3.9
26 14 52 58	12	4		−5 47.4	...	−52 17 7.1		0.474	...	−5.7
26 19 24 18	13	1	+0 3.84	...	19 54 18.37	...	9.910	...	+4.30	...
26 19 38 39	13	3	...	−1 57.1	...	−52 12 4.2	...	0.384n	...	−6.1
26 20 37 29	14	3	+0 2.57	...	19 53 28.78	...	9.904	...	+4.29	...

Greenwich Mean Time	* No.	No. of Comp.	$\Delta\alpha$	$\Delta\delta$	App. α	App. δ	log $p\Delta$ α	log $p\Delta$ δ	Red. to App. Pl. α	Red. to App. Pl. δ
1911 h m s			m s	′ ″	h m s	° ′ ″			s	″
Aug. 27 17 26 38	15	5	...	+1 52.2	.....	−51 47 36.5		9.531	...	−7.6
27 17 56 46	15	10	−0 27.82	.....	19 40 12.71	.....	9.867	..	+4.18	....
27 19 4 9	16	5	+0 30.22	.....	19 39 33.25	.....	9.904	..	+4.16	...
27 19 46 43	16	9	......	+1 56.4	....	−51 44 49.3	...	0.398n	..	−7.8

Mean Places for 1911.0 of the Comparison Stars.

*	α	δ	Authority
	h m s	° ′ ″	
No. 1	21 29 22.13	−52 11 59.4	Mag. $9\frac{3}{4}$. Micrometer Comparison with Star No. 2
2	21 28 50.67	−52 13 13.0	Mag. 6.8. Argentine General Catalogue No. 29.515
3	21 27 50	−52 6 59.4	Mag. 9.6. [C.P.D. −52° No. 11891 Mag. 9.6] mic. Comp. with Star No. 2
4	21 4 25.52	−52 39 14.1	Mag. 10. Micrometer Comparison with Star No. 5
5	21 5 3.66	−52 42 9.5	Mag. $7\frac{1}{4}$. General Argentine Catalogue No. 29016
6	20 59 42.26	−52 43 59.9	Mag. $8\frac{3}{4}$. General Argentine Catalogue No. 28874
7	20 35 54.42	−52 48 3.5	Mag. 9.0. Zone Catalogue Cordoba [20h No. 1100]
8	20 34 51.24	−52 47 32.8	Mag. $7\frac{3}{4}$. Argentine General Catalogue No. 28296
9	20 16 56.25	−52 37 54.5	Mag. 9.7. [C.P.D. −52° No. 11679 Mag. 10.2] Corboda meridian observ.
10	20 14 43.76	−52 37 44.9	Mag. $10\frac{1}{2}$. [C.P.D. −52° No. 11673 Mag. 10.0] mic. com. with Stars 9 and 11
11	20 12 32.22	−52 38 19.0	Mag. 9.8. [C.P.D. −52° No. 11664 Mag. 9.7] Cordoba meridian observ.
12	19 57 26	−52 11 14.0	Mag. 9.7. [C.P.D. −52° No. 11610 Mag. 9.6] Corcoba meridian observ.
13	19 54 10.23	−52 10 1.0	Mag. 11. Micrometer Comparisons with Star No. 14
14	19 53 21.92	−52 4 21.6	Mag. 9.4. [C.P.D. −52° No. 11598] Cordoba meridian observations
15	19 40 36.35	−51 49 21.1	Mag. 9.7. [C.P.D. −51° No. 11340 Mag. 9.2] Cordoba meridian observ.
16	19 38 58.87	−51 46 37.9	Mag. 9.8. [C.P.D. −51° No. 13331 Mag. 9.0] micrometer comp. with star 15

NOTES.

1911 Aug. 22. Comet pretty bright in the finder (about 4th magnitude) but very faint with the 12-inch. Almost round, about 15′ in diameter, more condensed in the middle. Atmospheric conditions bad.

24. At 13h 55m Greenwich M. T. the comet was still moving southwards. Comet appeared in the finder to be of 4.5 magnitude, but very faint in the main telescope. Nucleus comparable to a $10\frac{1}{2}$ magnitude star. Comet about 15′ in diameter in the finder. Seeing not very good.

1911 Aug. 25. Comet about $5\frac{1}{2}$-6th magnitude (in the finder), nucleus $10\frac{1}{2}$-11th magnitude. Comet in the finder about 15, in diameter. Sky good for the first observation, not so favorable for the others.

26. Comet is getting fainter. For the last observation measures in declination impossible. Comet invisible.

27. Comet very faint, probably 9-10th magnitude. For the last observation measures almost impossible, comet being too near horizon.

All the $\Delta\alpha$ are direct measures.

ON THE SECULAR VARIATIONS OF *JUPITER* AND *SATURN*,

By R. T. A. INNES.

In the *Transactions of the Royal Society of South Africa*, Vol. II, 1912. I showed that the relations given by JACOBI in his celebrated paper on the *Elimination of the Nodes in the problem of three Bodies* were closely fulfilled in the case of the Sun, *Jupiter* and *Saturn*, namely

$$\begin{array}{ccc} \textit{Jupiter} & & \textit{Saturn} \\ e\,\Delta\,e & = & -\,0.405\,e_1\,\Delta\,e_1 \\ \Delta\,i & = & +\,0.405\,\Delta\,i_1 \end{array}$$

in which e represents the eccentricity and Δe its instantaneous secular variation and Δi the instantaneous secular variation of the inclination to the invariable plane of the system.

In your *Journal* Nos. 656-657, Dr. G. W. HILL has a paper upon the *Secular Perturbations of the Four Outer Planets*, and it is natural to see how the figures he gives conform to the ideal case. Dealing firstly with the eccentricities, it will be sufficient if three dates only are used, viz., 1900 the mean, and 1000 and 2800 the means of the extreme dates. Then taking one century as the instantaneous unit, we have (from Dr. HILL, *A.J.*, 656, pp. 68-69).

Date	Δe	Δe_1
1000	+0.00017098	−0.00033066
1900	16424	34582
2800	15642	35970

whilst if we compute Δe from $[9.60769] \frac{e_1}{e} \Delta e_1$, we have

Date	Δe
1000	+0.0001686
1900	1620
2800	1543

The discrepancy is of the order of 1 to 2%.

Passing to the inclinations, the particular case asserts that the inclinations to the invariable plane increase or decrease together and, if we consider nearly circular orbits, this is easily seen to be true from first principles. If, however, we turn to page 62 of Dr. HILL's work, we see that the inclination of *Jupiter's* orbital plane is diminishing whilst that of *Saturn* is increasing. The contradiction is only apparent because Dr. HILL has referred the planets not to their own plane of greatest moments, but to the invariable plane of the four outer planets. Taking Dr. HILL's figures, I find the mutual inclination of the planes of *Jupiter* and *Saturn*, and their inclinations to their own plane of greatest moments to be

	Mutual Inclination	ι	ι_1
	° ′ ″	′ ″	′ ″
900	1 15 25.66	21 44.1	53 41.6
1900	1 15 20.57	21 42.0	53 38.6
2900	1 15 12.11	21 40.7	53 31.4

or the total change of $\iota = -3''.4$ and of $i_1 = -10''.2$ whilst $0.405\ \Delta i_1$ gives $-4''.1$ instead of $-3''.4$. But a more direct way of making the comparison is afforded by Dr. HILL's table of the positions of the planes with respect to another fixed plane on page 61. Thus the variations for *Saturn* are

$$+24''.1087\,T \quad -0''.21351\,T^2 \quad -0''.000652\,T^3$$
$$+33.2538\,T \quad +0.16374\,T^2 \quad -0.000851\,T^3$$

Those for *Jupiter* should be given by multiplying by -0.405. This is done in the following lines, Dr. HILL's figures being given immediately below:

$-9''.76\,T$	$+0''.0864\,T^2$	$+0''.000264\,T^3$	(-0.405)
-9.50	$+0.0862$	$+0.000244$	(HILL, p. 61)
-13.46	-0.0663	$+0.000345$	(-0.405)
-13.00	-0.0667	$+0.000321$	(HILL, p. 61).

The discrepancies are of the same order as for the eccentricities. JACOBI shows that the equations are rigorous in his ideal case, but LAPLACE had previously shown that approximate relations hold with all inequalities of long period (see BOWDITCH's translation, Vol. III, p. 318, etc.)

NOTE.

Misprints in Dr. HILL's paper p. 62.
Fourth line for 3 48 23.30 read 32.20.
Fourth line for 3 48 10.087 read 10.097.

Johannesburg, 1914, *January* 15.

OBSERVATIONS OF (624) HECTOR,

MADE WITH THE 26-INCH EQUATORIAL OF THE U. S. NAVAL OBSERVATORY.

BY H. E. BURTON, ASSISTANT IN THE OBSERVATORY.

[Communicated by Captain J. L. Jayne, U. S. Navy, Superintendent].

Date Wash. M.T.	*	Comp.	$\Delta\alpha$	$\Delta\delta$	App. α	App. δ	log $p\Delta$ α	log $p\Delta$ δ	Red. to App. Pl. α	Red. to App. Pl. δ
1913 h m s			m s	° ′	h m s	° ′ ″			s	″
Aug. 25 12 58 51	1	30 , 10	−1 35.47	+1 11.2	22 41 31.59	−10 30 21.1	8.822	0.822	+3.57	+17.8
31 11 14 22	2	30 , 10	+0 49.67	−3 40.8	22 38 18.57	−10 38 13.9	8.482*n*	0.824	+3.66	+17.6
Sept. 5 11 50 3	3	30 , 10	−1 42.58	−4 14.8	22 35 36.11	−10 44 31.5	8.427	0.825	+3.67	+17.7
9 10 46 53	4	30 , 10	+1 22.21	−5 5.2	22 33 29.45	−10 49 10.3	8.815*n*	0.824	+3.71	+17.4
25 10 18 50	5	30 , 10	−0 24.84	+4 23.4	22 25 41.55	−11 2 44.2	8.328	0.827	+3.71	+16.7
Oct. 3 10 55 49	6	30 , 6	−3 13.90	−2 13.1	22 22 28.73	−11 5 23.5	9.214	0.821	+3.65	+16.4

Mean Places of Comparison-Stars for 1913.0.

*	α	δ	Authority	*	α	δ	Authority
	h m s	° ′ ″			h m s	° ′ ″	
1	22 43 3.49	−10 31 50.1	A.G. Camb. U.S. 8020	4	22 32 3.53	−10 44 22.5	A.G. Camb. U.S. 7974
2	22 37 25.24	−10 34 50.7	A.G. Camb. U.S. 7999	5	22 26 2.68	−11 7 24.3	American Ephemeris, σ Aquarii
3	22 37 15.02	−10 40 34.4	A.G. Camb. U.S. 7998	6	22 25 38.98	−11 3 26.8	A.G. Camb. U.S. 7946

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PRELIMINARY RESULTS OF A SEARCH FOR PARALLELISM IN THE ORBIT-PLANES OF BINARY STARS,

By J. M. POOR.

The problem of the parallelism of orbits of double stars to some particular plane has interested a number of investigators, particularly EVERETT, SEE, DOBERCK, LEWIS and TURNER, and BOHLIN, whose studies have been based on the elements of the most accurately determined orbits. Miss EVERETT (*M. N.*, Vol. 56), from her results concludes that "it does not seem that any decided tendency on the part of the poles of the orbits to favor any special region of the sphere can reasonably be deduced." SEE (*Evolution of Stellar Systems*, Vol. I) concludes "that the orbits are not directly related to the Milky Way or any other fundamental plane of the heavens," and further that "even if confined originally to one plane the parallelism would have been disturbed by the action of foreign bodies during the ages required for the development of the visible universe." DOBERCK (*A. N.*, 3519) says, "I have for years past endeavored to find a law regulating the planes of the orbits, but have failed to do so, possibly because the orbits are distributed with reference to several different rules." Again (*A. N.*, 4291), he concludes that "the poles do not lie along the galaxy nor near the pole of the ecliptic. It appears to be more probable that they lie along the ecliptic than that they lie near the pole of the galaxy." LEWIS and TURNER (*M. N.*, Vol. 67) find "that orbits nearly at right-angles to the line of sight seem to avoid the galactic poles." On the whole (*Obsy.*, June, 1908), they concluded that the evidence favors a grouping of poles near the Milky Way. BOHLIN, on the contrary (*A. N.*, 4213), finds that the poles show a tendency to group themselves about two points; one near the pole of the galaxy and a second near the pole of the ecliptic and the apex of the Sun's way.

A solution of the problem based on the elements of known orbits presents difficulties because trustworthy orbits are few, and because to every real pole there corresponds a spurious pole. In two cases that of ζ *Cancri* and that of 42 *Comæ* (DOBERCK *loc. cit.*), the real and spurious poles are so near together that no practical difficulty exists, and in the case of *Sirius*, CAMPBELL (*Astroph. Jour.*, Vol. 21) finds the inclination of the orbit to be positive so that the spurious pole disappears.

At the meeting of the Astronomical and Astrophysical Society of America, held in Cleveland in December, 1912, the writer submitted a preliminary statement outlining a statistical method of investigating the parellelism of orbit-planes of binary stars to any particular plane in space. It was pointed out that were the orbit-planes of binary stars parallel, then because the apparent orbits of those situated on the great circle parallel to their orbit planes would be straight lines, while at the poles of this great circle the apparent orbits would be ellipses, the parallelism would show itself in a statistical study as a variation in correlation between position angle and distance of doubles in different parts of the sky.

As this problem of correlation is more easily solved in rectangular than in polar coördinates, position angle and distance may be replaced by rectangular coördinates $x = r \sin P$ and $y = r \cos P$, and the problem reduced to that of finding the correlation between x and y for all binary stars within a particular, limited part of the sky, and for every such part in the same manner, and finally making a study of the distribution of correlation coefficients on the celestial sphere. If parallelism to any circle exists it would be indicated by large values of the coefficient of correlation along this circle, except at points where the circle is nearly parallel to the x– or y– axis of the particular square, and smaller values as one approaches its poles where the correlation coefficient would become zero. Further the major axes of the ellipses of equal frequency would be parallel to the great circle in question. While the presence of optical doubles in the material studied might obscure, it was hoped that because of their chance distribution they might not entirely conceal the correlation between x and y in the case of physical binaries, if such correlation exists.

As a first step in undertaking this problem a card catalogue belonging to the Lund Observatory of all double stars in BURNHAM's *"General Catalogue,"* Vol. I, made at Lund under the direction of the writer at the suggestion of Professor CHARLIER for another purpose, was more fully developed. New doubles published by both ESPIN and AITKEN previous to 1913 and SCHEINER's list from the *Astrographic Catalogue* (Potsdam, Pub., Nr. 59), were added. Each card contained the catalogue number, right ascension, declination, magnitudes, position angle and distance of the double, and any other information likely to be of use. Observations and notes contained in BURNHAM's catalogue, Vol. II, were added whenever such additions seemed desirable. Upon completion of this catalogue the cards were separated into groups, each group including the stars in one of CHARLIER's forty-eight so-called "squares," described in *Medd. f. Lunds Astron. Obs.*, S. II, Nr. 8. The limits of the squares in declination are as follows:

A squares,	from	+90°	to	+66° 26′.6
B "	"	+66 26′.6	"	+30
C "	"	+30	"	0
D "	"	0	"	−30
E "	"	−30	"	−66 26′.6
F "	"	−66 26′.6	"	−90

The following table gives the limits in right ascension:

A_1	and F_1	from	0h.0	to	12h.
A_2	" F_2	"	12 .	"	24
B_1	and E_1	from	0h.0	to	2h.4
B_2	" E_2	"	2 .4	"	4 .8
B_3	" E_3	"	4 .8	"	7 .2
B_4	" E_4	"	7 .2	"	9 .6
B_5	" E_5	"	9 .6	"	12 .0
B_6	" E_6	"	12 .0	"	14 .4
B_7	" E_7	"	14 .4	"	16 .8
B_8	" E_8	"	16 .8	"	19 .2
B_9	" E_9	"	19 .2	"	21 .6
B_{10}	" E_{10}	"	21 .6	"	24 .0
C_1	and D_1	from	0h.	to	2.h
C_2	" D_2	"	2.	"	4.
C_3	" D_3	"	4.	"	6.
C_4	" D_4	"	6.	"	8.
C_5	" D_5	"	8.	"	10.
C_6	" D_6	"	10.	"	12.
C_7	" D_7	"	12.	"	14.
C_8	" D_8	"	14.	"	16.
C_9	" D_9	"	16.	"	18.
C_{10}	" D_{10}	"	18.	"	20.
C_{11}	" D_{11}	"	20.	"	22.
C_{12}	" D_{12}	"	22.	"	24.

The material studied is limited to thirty-six of the forty-eight squares, those lettered A, B, C, and D, the region completely covered by BURNHAM's catalogue.

To convert polar into rectangular coördinates the cards for a given square were arranged in order of increasing position angle and the quantities $x = r \sin P$ and $y = r \cos P$, found by the aid of GURDEN's *"Traverse Tables,"* were entered upon each card, except in the case of the A squares at the poles of rotation where, owing to the rapid convergence of the meridians, the position angle was corrected by means of tables computed for the purpose, so that x is measured along a system of "meridians" passing through the equinoxes and y is measured perpendicularly to these circles. The information thus entered upon the cards constituted the material from which the correlation between x and y could be found.

Experiment soon showed that some precepts for selecting the doubles to be studied must be laid down. Those finally adopted, put in the form of precepts for rejecting a pair were as follows:

(1) Doubles for which information was not sufficient for the purpose or appeared to be insufficiently accurate (*e. g.* HERSCHEL's doubles for which no recent observations were found) were rejected.

(2) All cases of doubles probably optical as determined from proper motion were rejected.

(3) Measures of multiple stars in the nature of the triangulation of a field were rejected. (*e. g.* θ *Orionis*).

(4) In case the brighter component of a pair was of the tenth magnitude it was excluded if its companion was as faint as the eleventh magnitude, and all cases in which the brighter star was fainter than the tenth magnitude were rejected. This in fact excludes but few pairs.

(5) All doubles separated by more than 100″ were finally rejected because it was found that for a given square such measures were in general scattering. As the work neared the end it became apparent that (5) was not sufficiently elastic. Its application was, however, continued for the sake of uniformity of treatment.

The question as to whether optical doubles were still to be found in the material selected in accordance with the above rules still remained, and after some preliminary computations it was decided to restrict the material used in the investigation in such way as to include only a limited number of probable optical doubles. Any method of selection would be open to question, but the following was finally employed:

We have (ANDRÉ, *"Traité d'Astronomie Stellaire," deuxieme partie*, p. 7), the familiar formula

$$\nu = \frac{N\,(N-1)}{8} \sin^2 1'' \times \beta^2$$

for the probable number of optical pairs within the distance β among N stars scattered at random over the en-

tire sphere. If we confine ourselves to a single square, this becomes with sufficient accuracy

$$\nu = 6 \sin^2 1'' \times \beta^2 N^2$$

If we put $N = a(1) + a(2) \quad * \quad * \quad * \quad + a(m)$, where $a(m)$ is the number of stars of magnitude m in a given square, we have an expression which on development, consists of terms of two kinds; one involving second powers and the other the products of the quantities $a(m)$.

These terms have the following forms

$$\nu' = 6 \sin^2 1''.\ \beta^2.\ a^2(m) \quad \text{and}$$
$$\nu'' = 12 \sin^2 1''.\ \beta^2.\ a(m)\ a(m')$$

The first expression gives the probable number of optical doubles in a given square among stars of magnitude m, whose separation is less than β and the second gives the probable number of optical doubles in a given square within the same limits of separation, one of whose components is of magnitude m and the other of magnitude m'. If, in this expression, we substitute for $a(m)$ and $a(m')$ their equivalents, as found by Charlier (*loc. cit.* p. 32)

$$a(m) = \frac{N}{k\sqrt{2\pi}}\, e^{-\frac{(m-m_0)^2}{2k^2}}$$

$$\text{we have } \nu' = \frac{3 \sin^2 1''}{\pi} \frac{N^2}{k^2} \beta^2 e^{-\frac{(m-m_0)^2}{k^2}}$$

$$\text{and } \nu'' = \frac{6 \sin^2 1''}{\pi} \frac{N^2}{k^2} \beta^2 e^{-\frac{(m-m_0)^2 + (m'-m_0^2)}{2k^2}}$$

For the regions studied by Charlier he found k nearly constant, its mean value being 3.038. Therefore, reducing the expressions to numbers and solving for β, we have

$$\log \beta = 5.854 - \log N + 0.0235\,(m - m_0)^2 + \log \sqrt{\nu'} \text{ and}$$
$$\log \beta = 5.870 - \log N + 0.0118\,[(m - m_0)^2 + (m' - m_0)^2] + \log \sqrt{\nu''}$$

If we put ν' and ν'' equal to unity, we have expressions which give the value of β corresponding to a single optical pair in the particular case when we know N and m_0. The resulting values of β have the convenience that they can be easily modified if it is desired to admit z optical doubles in a given case, for this can be done by simply taking $\beta\sqrt{z}$ as the limit in place of β. The material used in this study was selected by putting $\nu' = \nu'' = 1$. Charlier gives the values of N and m_0 for nine of the C-squares, and data from which these quantities may be determined for the remaining C-squares. With these values the quantities β were found for all combinations of magnitude up to fourteen, except in the case of stars brighter than magnitude six, which, because their numbers were small, were grouped together. In these cases β was determined by solving the formulas expressed directly as functions of the number of stars of a given magnitude. In making the computations the value m employed was taken as half-way between two successive integers and in application all stars whose magnitudes were between m and $m + 0.99$ were included in the magnitude designated as m where m is an integer.

The values of β thus found for the various combinations of magnitude of the stars in a given square were called "β-curves" for the particular square, and their application consisted in classifying the doubles of a square according to magnitude of the principal star and then subdividing each class according to magnitude of the companion. Only those pairs were retained whose separation was less than the value of β as found in the table for the case in question unless, as occasionally happened, common proper motion indicated physical connection.

The constants of the curves are based on the Harvard scale, the "curves" are but isolated points and their application assumes a normal distribution of the stars of each magnitude which forms a distinct class. An attempt was made to reduce the magnitudes to the Harvard scale, but it had to be given up, and in the absence of better information as to magnitudes, those in Burnham's catalogue, improved when necessary, if possible, were employed. This method of selecting pairs for study has defects, yet it has the obvious advantage of *standards dependent on star density*, and eliminates personality in an attempt to reach those doubles which are most likely to be physically connected.

The curves were applied to the particular squares from which they were derived, but in the case of the D-squares, as no information in regard to N and m_0 was at hand, the curves for a C-square were applied to the D-square diametrically opposite (*i. e.*, to the square having the same galactic latitude).

To construct curves for the A- and B- squares, the β-curves already computed for the C-squares were divided into four groups of three each, those most nearly alike being placed in the same group. By averaging the values of β for a given combination of magnitudes within a group, "normal β-curves" were obtained, which are given below in Tables I, II, III, IV. These tables are tables of double entry in which β in a given case is found at the intersection of the column giving m with the line giving m' or m in case both stars are of the

Table I.

m' \ m	m<6	6	7	8	9	10
7	...	...	80″	...	...	...
8	...	59″	31	24″	...	...
9	62″	34	18	9.8	8.0″	...
10	37	21	11	6.0	3.4	2.9″
11	24	13	6.8	3.7	2.1	...
12	16	8.4	4.5	2.5	1.4	...
13	10.7	5.9	2.9	1.7	1.0	...
14	7.4	4.2	2.2	1.2	0.7	...

TABLE II.

m' \ m	$m<6$	6	7	8	9	10
8		94″	49″	39″	. . .	. . .
9		56	29	17	14″	. . .
10	71″	36	19	10.4	6.2	5.6″
11	48	24	13	7.0	4.4	. . .
12	34	17	8.9	5.0	3.0	. . .
13	26	13	6.8	3.8	2.3	. . .
14	21	11	5.5	3.1	1.8	. . .

TABLE III.

m' \ m	$m<6$	6	7	8	9	10
8	. . .	. . .	71″	58″	. . .	. . .
9	. . .	78″	43	25	21″	. . .
10	94″	50	28	16	9.7	8.7″
11	64	34	19	11	6.6	. . .
12	46	24	13	7.7	4.6	. . .
13	34	18	10	5.8	3.5	. . .
14	27	14	7.9	4.6	2.7	. . .

TABLE IV.

m' \ m	$m<6$	6	7	8	9	10
8	. . .	. . .	85″	71″	. . .	. . .
9	. . .	94″	53	32	28″	. . .
10	. . .	63	36	21	14	12.8″
11	83″	44	25	15	9.6	. . .
12	61	33	19	11	7.0	. . .
13	48	26	15	8.7	5.5	. . .
14	40	21	12	7.2	4.6	. . .

same magnitude. I was applied to squares whose centers are between 0° and 15° galactic latitude; II to those between 15° and 30°; III to those between 30° and 55°; and IV to those above 55°. Cases for which β exceeds 100″ do not appear in these tables.

The material thus prepared and selected for each of the thirty-six squares was distributed in correlation tables for which the "class breadth" was 5″, each pair being entered twice, once with the signs of x and y as found by computation and again with signs reversed to overcome ambiguity in regard to position angle in the case of pairs whose components are of the same magnitude. This has the effect of making the probability curves in x and y symmetrical and introduces some checks on the computations. The characteristics of the probability curves were computed and checked by the formulæ to be found in *Medd. f. Lunds Astron. Obs.*, S. II, Nr. 4, and the correlation coefficients were found by familiar methods. Table V gives the characteristics of the correlation surfaces for the various squares. N is double the number of pairs in each square. σ_x and σ_y are the "dispersions" or "standard deviations" for the x- and y- curves respectively; e_x and e_y are the "excess" which in these results are always positive indicating an abnormal distribution in the material studied, a matter to which it is hoped to give more study at a later time; r is the coefficient of correlation, and p is the position angle of the major-axis of the ellipse of equal frequency, positive when reckoned from north toward the east and negative when reckoned from the north toward the west, so that it is always numerically less than 90°. It is determined from the expressions

$$\tan 2\phi = \frac{2\sigma_x\,\sigma_y r}{\sigma_x^2 - \sigma_y^2} \qquad p = 90^\circ - \phi$$

TABLE V.

Square	N	σ_x	σ_y	e_x	e_y	r	p
A_1	470	+2.16	+1.96	+1.13	+0.95	+0.024	+83
A_2	478	1.07	2.12	0.33	0.88	+0.128	+15
B_1	994	1.20	1.32	1.54	1.45	+0.021	+ 7
B_2	788	1.67	1.74	0.78	1.23	+0.110	+36
B_3	766	1.79	1.39	2.18	1.58	−0.024	−87
B_4	664	2.34	2.27	0.98	0.03	−0.039	−62
B_5	434	1.87	1.47	0.61	0.93	−0.018	−88
B_6	456	2.20	1.75	0.90	0.64	+0.334	+62
B_7	556	1.70	1.64	0.94	0.41	−0.097	−56
B_8	826	1.64	2.06	0.41	1.22	+0.058	+ 7
B_9	1276	1.87	1.66	1.50	1.13	−0.121	−67
B_{10}	1102	1.56	1.80	0.73	0.94	−0.120	−20
C_1	498	1.68	1.16	1.10	0.38	+0.069	+85
C_2	562	2.01	1.92	0.61	0.70	−0.020	−77
C_3	648	2.03	2.14	0.70	0.76	+0.104	+31
C_4	478	1.60	1.78	1.15	0.80	−0.091	−20
C_5	364	2.43	2.13	0.63	0.32	+0.011	+88
C_6	372	1.70	2.97	0.47	0.44	+0.132	+ 6
C_7	424	2.45	2.25	0.89	0.61	+0.124	+62
C_8	476	1.75	1.36	1.06	0.50	+0.283	+66
C_9	538	2.01	1.87	0.91	0.61	+0.220	+54
C_{10}	806	1.65	1.07	2.78	1.69	−0.007	−90
C_{11}	666	1.04	1.21	0.32	1.07	−0.127	−20
C_{12}	486	1.36	2.12	0.53	1.48	−0.343	−19
D_1	358	2.41	2.44	0.80	0.49	−0.338	−44
D_2	358	2.11	1.27	0.78	0.22	+0.224	+78
D_3	494	1.98	2.19	0.52	1.09	+0.078	+19
D_4	542	1.41	1.27	1.26	1.45	+0.019	+85
D_5	302	1.71	2.06	0.99	0.96	+0.072	+11
D_6	308	1.96	1.82	1.44	0.71	+0.252	+53
D_7	298	1.72	1.79	0.27	0.20	−0.284	−41
D_8	392	1.86	1.78	0.40	0.49	−0.164	−52
D_9	440	1.31	1.56	0.41	0.53	+0.060	+ 9
D_{10}	452	1.40	1.49	0.63	0.53	+0.086	+27
D_{11}	450	1.41	1.86	0.26	0.66	+0.028	+ 3
D_{12}	362	2.16	1.75	0.83	0.52	−0.399	−59

σ_x, and σ_y are in units of the class breadth, (*i. e.* 5″).

To test the results of Table V supplementary studies of four squares A_1, A_2, B_3, and B_9 were carried along with those made under uniform conditions described above. For each square the limiting value of β was increased to 3β except in the case of pairs composed of stars of magni-

tude 9 or 10, in squares B_3 and B_9. These squares include the greater portion of the material in SCHEINER's list which consists largely of faint pairs of wide separation, which had been eliminated by the standards set up, and as it was desirable to include them in the investigation the limit for $m = 9$, $m' = 10$ was taken at $25''$ (7β), and for $m = m' = 10$ at $19''$ (6β). The outside limit of $100''$ was retained. Further studies were also made in the case of these two squares by rejecting from the correlation tables, from which the results of Table V are found, all pairs whose separation is greater than $60''$. The results of these last two computations, marked with an asterisk, are given in Table VI together with the results of the first computations just described.

TABLE VI.

Square	N	σ_x	σ_y	e_x	e_y	r	p
A_1	548	+2.71	+2.53	+0.56	+0.43	+0.098	+63°
A_2	580	2.01	2.54	+0.17	+0.52	+0.052	+ 6
B_3	1586	2.09	1.93	+0.79	+0.34	±0.000	±90
*B_3	758	1.10	1.00	−0.08	−0.07	−0.006	0
B_9	2964	2.23	2.12	+0.58	+0.48	−0.003	−88
*B_9	1260	1.29	1.33	+1.01	+1.22	−0.132	−59

In A_1, A_2, and B_9, the changes in the positions of the axes of the ellipses are certainly not greater than must be expected until better methods of selecting the material can be devised, but in B_3 the position of the axis is indeterminate a result, for which an explanation later appears. Later computations in this paper are, however, based on the results in Table V.

Though it had been anticipated that parallelism would be indicated but roughly, if at all, the results (Table V) as tabulated at least, seemed to lead to the conclusion that either no parallelism exists or else that the method of selection of material must be modified. When, however, the correlation coefficients were plotted to scale in the direction indicated by the angle p upon a globe properly divided into squares there was found, with four distinct exceptions (A_2, B_8, D_7, and D_8), a rough parallelism to a circle whose pole was estimated to be at R. A. 80° to 90° and Decl. +20° to +35° together with the fact that the correlation coefficients follow roughly the conditions above set forth for parallelism, though the northern pole was more clearly marked than the southern.

As soon as this graphical solution had been made, it was recognized that this pole lies near the "vertex of preferential motions of the stars," a result indicating that a "preferential pole" for the orbit-planes of binary stars lies near the vertex of preferential motions of the stars.

Several graphical representations of the results were unsuccessfully attempted, but on advising with Professor C. N. HASKINS of the Mathematical Department of the College in regard to the matter, he suggested the application of the following analytic method for the determination of the position of the pole of this plane which has been employed in place of a graphical solution, and which is essentially that employed by NEWCOMB in his determination of the galactic plane (CARNEGIE Pub., No. 10).

If l, m, and n are the direction cosines of the major-axis of the ellipse of equal frequency on the surface of the sphere at the middle point of the meridian passing through the center of a square and λ, μ and ν the direction cosines of the normal to the plane to which it is parallel we have the conditions $l\lambda + m\mu + n\nu = 0$. But as no one plane satisfies this condition for every square, the solution requires that the expression $\Sigma\,(l_i\,\lambda + m_i\,\mu + n_i\,\nu)^2$ be minimized subject to the condition that $\lambda^2 + \mu^2 + \nu^2 = 1$ which leads at once to the cubic equation used by NEWCOMB. Computing l, m, and n for each square and giving equal weight to each result we have the cubic equation

$$X^3 - 36X^2 + 423.8X - 1625.1 = 0$$

which has the roots

$$X_1 = +\ 8.9$$
$$X_2 = +12.5$$
$$X_3 = +14.6$$

or rejecting the results for A_2, B_8, D_7, and D_8, we have the equation

$$X^3 - 32X^2 + 326.2X - 1033.5 = 0$$

with the roots

$$X_1 = +\ 6.26$$
$$X_2 = +12.2$$
$$X_3 = +13.6$$

Employing the smallest root in each case we have by NEWCOMB's formulæ when all the results are taken into account $\alpha = 70°$, $\delta = +24°$ and when the results of the four squares A_2 B_8 D_7 D_8 are rejected $\alpha = 80°$, $\delta = +29°$ pointing with unexpected accuracy to the vertex of preferential stellar motions for which CAMPBELL (*Stellar Motions*, page 147) gives a mean position $\alpha = 93°$, $\delta = +12°$.

The location of the pole makes possible an explanation of the behavior of the characteristics of the surfaces in B_3 and B_9, as B_3 lies near the pole, *i. e.* where the distribution should be circular, and B_9 is near the great circle, *i. e.*, where the distribution should be elliptical.

The results in A_2, B_8, D_7, and D_8, are singular in that they form two pairs of adjoining squares widely separated on the sphere for which the characteristics, though not in harmony with those of adjoining squares, are yet similar for the same pair.

Effort has not been spared to secure numerical accuracy

as far as possible or repeat the earlier computations in which there were departures from the standards set up for determination of the quantities in Table V, but as these results are but preliminary and from the nature of the material on which they are based must be regarded from the view point of their general tendency rather than that of accuracy of a particular result no "weighted" pole has been determined at this time.

The position of the pole finds a singular support in the fact that the three binaries for which the orbit planes may be regarded as known, namely, *Sirius*, 42 *Comæ*, and *ξ Cancri* have their orbit poles (Everett, *loc. cit.*) within about 30° of the vertex of stellar motions. The conclusion of Lewis and Turner also, that orbits perpendicular to the line of sight avoid the galactic poles, points near which the circle of parallelism here found passes, is not without interest in this connection.

The results of this study indicate a tendency toward perpendicularity between preferential motions and orbit planes. Whether there exist general tendencies toward perpendicularity between orbit planes and directions of motion in the case of systems moving at considerable angles with the preferential directions as indicated by the case of the solar system and Bohlin's suggestion (*loc. cit.*) in regard to the doubles of his "group II" (and the great nebula in *Andromeda*) must remain a matter for future study. Unfortunately for determination of parallax in special cases individual deviations from such perpendicularity whether from disturbances as suggested by See or other cause, are certainly large.

Though the region about the south pole has not yet been studied, the completion of a preliminary survey of material of Burnham's catalogue, covering three-fourths of the sky by uniform methods marks the end of a stage in the work where it seems desirable to make a statement as to methods employed and results obtained before going on with a matter which is subject to the delays incident to carrying on such computations single-handed, save for such assistance as can be rendered by untrained computers.

Shattuck Observatory, Hanover, N. H., March 25, 1914.

OBSERVATIONS OF ASTEROIDS,

MADE WITH THE 26-INCH EQUATORIAL OF THE U. S. NAVAL OBSERVATORY.

By C. B. WATTS, Assistant in the Observatory.

[Communicated by Captain J. L. Jayne, U. S. Navy, Superintendent.]

Date Wash. M.T.	✱	Comp.	Δα	Δδ	App. α	App. δ	log p Δ α	log p Δ δ	Red. to App. Pl. α	Red. to App. Pl. δ
				(55) *Pandora.*						
1913 h m s			m s	′ ″	h m s	° ′ ″			s	″
Sept. 11 11 19 24	1	30, 10	−0 57.65	−1 32.4	0 38 6.53	+1 48 41.1	9.347*n*	0.725	+3.43	+24.0
22 13 35 6	2	30, 6	+1 52.07	−8 25.7	0 28 56.66	+1 31 21.1	9.151	0.726	+3.56	+24.3
24 12 52 37	3	30, 6	+3 37.18	−6 59.3	0 27 10.29	+1 27 44.4	8.892	0.726	+3.59	+24.4
Oct. 12 9 9 17	4	30, 6	+1 52.36	+3 21.6	0 11 16.32	+0 57 45.8	9.277*n*	0.732	+3.67	+24.3
26 10 6 5	5	25, 5	−3 13.78	+4 50.2	0 1 50.13	+0 50 49.4	8.675	0.732	+3.64	+24.0
				(415) *Palatia.*						
Oct. 4 11 40 23	6	30, 6	−1 42.88	+8 59.1	0 58 10.81	− 8 29 30.3	8.698*n*	0.809	+3.74	+25.7
Nov. 7 9 30 21	7	30, 6	+0 58.74	−7 36.5	0 37 32.47	−10 33 29.4	6.960*n*	0.823	+3.82	+22.3
Estimated magnitude 10.0 on Oct. 4.										
				[1913 *SP*].						
Oct. 5 13 9 23	8	30, 10	+0 48.57	−5 54.9	0 5 37.88	− 1 19 44.6	9.362	0.751	+3.67	+23.9
Estimated magnitude 12.5.										
				(104) *Clymene.*						
Oct. 22 8 51 49	9	30, 6	−1 21.30	+1 19.2	1 3 39.28	+ 5 0 40.7	9.387*n*	0.696	+3.78	+26.2
				(127) *Johanna.*						
Oct. 31 9 32 36	10	30, 6	+3 52.38	−3 17.5	0 52 18.24	+ 3 17 56.6	8.906*n*	0.708	+3.74	+25.6
Nov. 2 9 43 34	10	30, 6	+2 26.23	−5 34.6	0 50 52.10	+ 3 15 39.5	8.606*n*	0.708	+3.75	+25.6
4 9 30 39	11	25, 5	+4 29.57	−4 36.7	0 49 31.28	+ 3 13 51.0	8.679*n*	0.708	+3.75	+25.6
6 9 2 38	11	30, 6	+3 13.94	−2 56.1	0 48 15.64	+ 3 12 31.5	8.931*n*	0.709	+3.74	+25.5
17 8 13 15	12	30, 6	+1 9.54	−3 37.3	0 42 51.65	+ 3 14 16.1	8.938*n*	0.708	+3.68	+25.0
21 7 26 43	13	30, 6	+1 10.39	+4 33.9	0 41 37.31	+ 3 18 50.1	9.158*n*	0.709	+3.66	+24.8
Estimated magnitude 10.5 on Oct. 31.										

Mean Places of Comparison-Stars for 1913.0.

*	α	δ	Authority	*	α	δ	Authority
	h m s	° ′ ″			h m s	° ′ ″	
1	0 39 0.75	+ 1 49 49.5	A.G. Albany 171	8	0 4 45.64	− 1 14 13.6	A.G. Nicolajew 13
2	0 27 1.03	+ 1 39 22.5	A.G. Albany 96	9	1 4 56.80	+ 4 58 55.3	A.G. Albany 310
3	0 23 29.52	+ 1 34 19.3	A.G. Albany 85	10	0 48 22.12	+ 3 20 48.5	A.G. Albany 222
4	0 9 20.29	+ 0 53 59.9	A.G. Nicolajew 21	11	0 44 57.96	+ 3 15 2.1	A.G. Albany 209
5	0 5 0.27	+ 0 45 35.2	A.G. Nicolajew 15	12	0 41 38.43	+ 3 17 28.4	B.D. +2°100 compared with A.G. Albany 188 $\Delta\alpha = +0^m34^s.20$, $\Delta\delta = +1'43''.1$ Jan. 6, 1914
6	0 59 49.95	− 8 38 55.1	A.G. Wien-Ottak. 215				
7	0 36 29.91	−10 26 15.2	A.G. Camb. U.S. 128	13	0 40 23.26	+ 3 13 51.4	A.G. Albany 177

NOTE.

Asteroids (55), (104), and (127) were found photographically by G. H. PETERS, Assistant in the Observatory. Asteroids (415) and [1913 *S P*] were found photographically by J. H. METCALF.

OBSERVATIONS OF COMET *c* 1913 (*ZINNER-GIACOBINI*),

MADE WITH THE 12-INCH EQUATORIAL AT THE NATIONAL ARGENTINE OBSERVATORY, CORDOBA,

BY A. ESTELLA GLANCY.

1913 Greenwich M.T.	*	Comp. α	Comp. δ	$\Delta\alpha$	$\Delta\delta$	App. α	App. δ	log $p\Delta$ α	log $p\Delta$ δ	Red. to App. Pl. α	Red. to App. Pl. δ
h m s				m s	′ ″	h m s	° ′ ″			s	″
Nov. 29 13 46 18	1	8*t*	8	−1 31.26	+6 23.3	22 44 4.02	−36 20 37.6	9.676	9.899*n*	+3.69	+ 5.9
Dec. 1 13 56 31	2	10	10	+ 38.16	−3 38.0	22 58 47.09	−37 0 33.3	9.685	9.885*n*	+3.74	+ 6.5
8 14 17 39	3	10	9	− 19.35	−9 3.0	23 49 51.89	−38 15 30.2	9.689	9.770*n*	+3.80	+ 8.4
15 14 57 22	4	9	10	− 18.23	−7 36.0	0 35 24.42	−38 0 35.6	9.719	9.986*n*	+3.82	+10.0
26 13 28 25	5	10	10	− 43.78	−3 4.1	1 33 44.68	−35 42 15.3	9.481	9.069	+3.76	+10.9

t signifies transits. Observations taken on Nov. 30 and Dec. 2, will be published later after the positions of the comparison stars have been determined.

Mean Places of Comparison Stars for 1913.0.

*	α	δ	Authority	*	α	δ	Authority
1	22 45 31.59	−36 27 6.8	Argentine Gen. Cat. 31089	4	0 35 38.83	−37 53 9.6	Argentine Gen. Cat. 600
2	22 58 5.19	−36 57 1.8	Argentine Gen. Cat. 31310	5	1 34 24.70	−35 39 22.1	Argentine Gen. Cat. 1584
3	23 50 7.44	−38 6 35.6	Argentine Gen. Cat. 32257				

Cordoba, February 12, 1914.

ELEMENTS AND EPHEMERIS OF COMET *f* 1913 (*DELAVAN*)

BY P. F. DELAVAN AND B. H. DAWSON.

The following elements were computed from six observations at La Plata combined into three places. The observations are December 17, HUSSEY; December 17, DAWSON; December 30, DAWSON; December 30, HUSSEY; and January 8, DAWSON.

ELEMENTS.

T = 1914 October 30.07304. Gr. M.T.

$$\left.\begin{aligned} \omega &= 97^\circ\ \ 4'\ 22''.1 \\ \Omega &= 58^\circ\ 43'\ 28''.0 \\ i &= 69^\circ\ \ 4'\ 12''.9 \end{aligned}\right\}\ 1914.0$$

$$\log q = 0.0524510$$

Residuals (O–C); $\Delta\lambda \cos\beta = -\ 5''.7$

$\Delta\ \beta = -\ 1''.3$

CONSTANTS FOR THE EQUATOR OF 1914.0.

$$x = r[9.7797956]\ \sin\ (217^\circ\ 31'\ 56''.4 + v)$$
$$y = r[9.9082600]\ \sin\ (201^\circ\ 29'\ 22''.1 + v)$$
$$z = r[9.9960211]\ \sin\ (117^\circ\ \ 8'\ 53''.4 + v)$$

Ephemeris.

1914 Gr.M.T.	α	δ	log. Δ	Mag.
	h m s	° ′ ″		
Mar. 1.5	2 42 20.1	+ 3 44 34	0.5904	9.1
5.5	2 43 47.0	4 28 24	0.5919	. .
9.5	2 45 26.7	5 12 32	0.5932	.
13.5	2 47 19.1	5 56 58	0.5942	.
17.5	2 49 23.4	6 41 41	0.5950	.
21.5	2 51 39.7	7 26 40	0.5954	9.0
25.5	2 54 7.4	8 11 56	0.5955	.
29.5	2 56 46.1	8 57 28	0.5953	.
Apr. 2.5	2 59 35.3	9 43 16	0.5948	8.9
6.5	3 2 34.7	10 29 20	0.5939	.
10.5	3 5 44.2	11 15 42	0.5927	.
14.5	3 9 3.4	12 2 23	0.5911	8.8
Apr. 18.5	3 12 32.3	12 49 25	0.5891	
22.5	3 16 10.8	13 36 49	0.5867	8.7
26.5	3 19 58.7	14 24 37	0.5841	.
30.5	3 23 56.0	15 12 51	0.5810	8.6

Magnitudes are based on an assumption of $9^m.3$ on December 30.5.

The ratio of the residuals indicates that they can not be materially reduced on the hypothesis of parabolic motion.

These elements indicate that the comet will rise a little brighter than fifth magnitude near perihelion.

NEUJMIN'S COMET,

By F. E. SEAGRAVE.

The elliptic elements of Neujmin's Comet have been computed from three normal positions based upon observations taken at the Roman College and at Copenhagen.

t = 250.442259	λ = 357° 32′ 59″.66	β = + 0° 57′ 50″.31
t' = 278.429111	λ' = 358 11 53 .13	β' = +12 3 19 .05
t'' = 301.304204	λ'' = 0 30 2 .90	β'' = +16 3 55 .17
$\odot$ = 164° 38′ 0″.52	Log R = 0.0031689	Ψ = 161° 44′ 7″.49
$\odot'$ = 192 1 19 .73	Log R' = 9.9998087	K = 357 22 0 .79
$\odot''$ = 211 43 49 .31	Log R'' = 9.9970253	β_0 = 4 22 16 .572

The following are the results from the fifth and last hypothesis:

ζ = 9° 48′ 16″.04	z' = 10° 59′ 11″.36
Log ρ = 9.7427944	Log r = 0.1907179
Log ρ' = 9.8127787	Log r' = 0.2158434
Log ρ'' = 9.9201215	Log r'' = 0.2458493
l = 349° 12′ 15″.80	b = +0° 20′ 37″.34
l' = 6 34 57 .73	b' = +4 50 34 .46
l'' = 19 10 23 .49	b'' = +7 49 5 .44
u = 1° 20′ 38″.63	Log p = 0.4330720
u' = 19 16 24 .56	Log p = 0.4330716
u'' = 32 8 4 .06	Log p = 0.4330719
v = 15° 4′ 1″.33	E = 5° 24′ 25″.50
v' = 32 59 47 .26	E' = 12 4 25 .44
v'' = 45 51 26 .76	E'' = 17 10 41 .12
M = 1° 13′ 44″.53	M_0 = 11° 40′ 10″.72
M' = 2 47 58 .27	M_0 = 11 40 10 .73
M'' = 4 4 59 .32	M_0 = 11 40 10 .72

Time perihelion August 16, 1913.

E = March 12.50000 = 1914. G.M.T.
M = 11° 40′ 10″.72
ω = 346° 16 37 .30
π = 334 10 55 .32
Ω = 347 54 18 .02
i = 14 49 2 .63
Log e = 9.8886549
Log a = 0.8297506
Log q = 0.1841556
μ = 202″.0136
P = 17.5644 years.

$$x = r\,(9.9993755)\ \sin(78°\ 17'\ 45''.46 + u)$$
$$y = r\,(9.8971762)\ \sin(345\ 54\ 8\ .84 + u)$$
$$z = r\,(9.7899284)\ \sin(352\ 13\ 32\ .96 + u)$$

Check.

$$q = p/(1+e) = a\,(1-e) = (a\cos\varphi)\tan\ (45° - \tfrac{1}{2}\varphi)$$

0.4330716	0.8297506	0.6314111
0.2489160	9.3544049	9.5527448

Log q = 0.1841556 = 0.1841555 = 0.1841559 = Log q

CONTENTS.

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THE MOON'S MEAN LONGITUDE, 1908–13, AND THE ECLIPSE OF AUGUST 21, 1914,

By FRANK E. ROSS.

[Communicated by Captain J. A. Hoogewerff, U.S. Navy, Superintendent U. S. Naval Observatory.]

An extended inquiry into the motion of the Moon was completed by Simon Newcomb in 1908, in which the observational data were occultations, almost exclusively. In his investigation, comparison between theory and observation was carried to August, 1908. At the suggestion of the writer, the Director of the *Nautical Almanac*, Professor W. S. Eichelberger, agreed that an extension of this comparison to date was desirable. The work was undertaken by the *Almanac* office and put in charge of the writer, with such assistance from the other members of the office force as might be needed. Most of the computations were duplicated by Mr. Alfred Doolittle.

The occultations used in the present comparison have been carefully selected from among those observed at the U. S. Naval Observatory since 1908. The bulk of those selected were observed with both the twenty-six inch and twelve-inch equatorials. This is especially true of bright limb occultations. Only occultations giving rise to a large coefficient in longitude, E, were chosen, which means that the occultations were central. The correction to the Moon's mean longitude, δM, is given by

$$E \,.\, \delta M = s' - D\,,$$

where s' is the apparent semi-diameter of the Moon, and D the tabular distance between the Moon's center and the occulted star, computed for the instant of disappearance or reappearance. This equation virtually assumes that the mean longitude is the only element needing correction. The tabular theory has been made to include the corrections to the remaining elements computed by Newcomb, as explained below.

Table I.

Date	Star	Phen.	Tabular Correction to Hansen-Newcomb $\delta\lambda$	$\delta\beta$	$\delta\pi$	E	$s'-D$	v
			″	″	″		″	″
1908 Apr. 13	ν *Virginis*	DD	+ 6.87	− .24	+.41	−0.93	−2.5	−1.8
June 11	o' *Libra*	DD	+ 9.13	− .61	.45	−1.00	−1.4	−0.7
Aug. 9	χ^3 *Sagittarii*	DD	+ 8.20	+ .40	.49	−0.89	+0.6	+1.2
Aug. 10	*Pi* 20.146	DD	+ 7.31	+ .47	.50	−1.07	−0.1	+0.6
Aug. 10	*Pi* 20.146	RB	+ 7.31	+ .47	.50	+1.11	−3.8	(−4.6)
Dec. 8	9 *Geminorum*	RD	+ 6.61	+1.06	.44	+0.97	−0.4	−1.1
1909 Feb. 28	8 *Geminorum*	DD	+ 6.13	+ .46	.44	−0.95	−0.5	+0.2
Mar. 14	63 *Ophiuchi*	RD	+ 8.39	+ .61	.46	+1.00	+1.9	+1.1
Mar. 28	ω *Geminorum*	DD	+ 6.17	+ .51	.44	−0.94	−1.6	−0.8
Mar. 28	ω *Geminorum*	RB	+ 6.17	+ .51	.44	+0.94	0.0	(−0.8)
Apr. 24	52 *B. Geminorum*	DD	+ 6.21	+ .62	.45	−0.98	−0.7	+0.1
July 10	ξ' *Ceti*	DB	+ 0.59	+ .21	.49	−1.00	+1.0	+1.8
July 10	ξ' *Ceti*	RD	+ 0.59	+ .21	.49	+1.06	−0.8	−1.6
July 27	84 *B. Scorpii*	DD	+ 9.30	−0.11	.43	−0.95	+0.8	+1.6
Aug. 29	154 *B. Capricorni*	DD	+ 7.38	+1.25	.48	−1.10	−1.1	−0.2
Sept. 29	26 *Ceti*	RD	+ 5.39	+ .92	.50	+1.16	+2.7	+1.8
Nov. 19	37 *Capricorni*	DD	+ 9.13	+1.54	+.48	−0.78	−2.0	−1.4

TABLE I. — *Cont.*

Date	Star	Phen.	Tabular Correction to Hansen–Newcomb δλ	δβ	δπ	E	s − D	v
			″	″	″		″	″
1910 May 19	46 *Virginis*	DD	+ 7.73	+0.31	+.40	−0.51	−0.8	−0.3
July 24	*τ Aquarii*	RD	+ 4.87	+1.46	.17	+0.76	+1.8	+0.9
Oct. 24	*ν′ Cancri*	DB	+ 3.44	+0.33	.47	−0.71	+0.2	+0.8
Oct. 24	*ν′ Cancri*	RD	+ 3.44	+0.33	.47	+0.79	+1.7	+1.0
Nov. 17	315 *B. Tauri*	RD	+ 5.15	+0.64	.50	+1.14	+1.8	+0.7
Dec. 14	192 *B. Tauri*	DD	+ 7.59	+1.04	.50	−1.04	+1.1	+2.1
Dec. 17	*c Geminorum*	DB	+ 5.54	+0.78	.48	−1.06	−0.4	+0.7
Dec. 17	*c Geminorum*	RD	+ 5.54	+0.78	.48	+0.90	+1.0	+0.1
1911 Jan. 5	*ψ³ Aquarii*	DD	+ 8.12	+1.79	.46	−0.64	+0.1	+0.7
Jan. 5	*ψ³ Aquarii*	RB	+ 8.12	+1.79	.46	+0.90	−0.8	(−1.7)
Jan. 9	*o Arietis*	DD	+ 8.85	+1.23	.50	−0.92	+0.1	+1.0
Jan. 9	*o Arietis*	RB	+ 8.85	+1.23	.50	+1.03	+0.1	(−0.9)
Feb. 10	*c Geminorum*	DD	+ 8.06	+0.62	.49	−1.03	−1.3	−0.2
Feb. 10	*c Geminorum*	RB	+ 8.06	+0.62	.49	+0.86	+0.7	(−0.2)
Aug. 9	37 *Capricorni*	DB	+ 6.52	+1.46	.42	−0.88	−3.6	−2.3
Aug. 9	37 *Capricorni*	RB	+ 6.52	+1.46	.42	+0.94	+1.0	(−0.4)
Sept. 6	161 *B. Capricorni*	DD	+ 7.08	+1.77	.42	−0.96	−2.2	−0.7
Sept. 6	161 *B. Capricorni*	RB	+ 7.08	+1.77	.42	+0.92	−2.3	(−3.8)
1912 Jan. 27	45 *Arietis*	DD	+10.02	+1.07	.48	−0.99	−2.8	−0.7
Apr. 4	169 *B. Libra*	DB	+ 8.28	+1.41	.43	−0.96	−4.2	−1.9
Apr. 4	169 *B. Libra*	RD	+ 8.28	+1.41	.43	+0.86	+1.3	−0.8
May 29	31 *B. Scorpii*	DD	+ 9.33	+0.93	.43	−0.95	−1.3	+1.1
Aug. 3	3 *Piscium*	DB	+ 6.38	+1.04	.44	−0.62	−1.5	+0.2
Aug. 3	3 *Piscium*	RD	+ 6.38	+1.04	.44	+0.78	+3.9	+1.7
Sept. 5	49 *Auriga*	DB	+ 5.87	−0.13	.49	−0.89	−2.7	−0.2
Sept. 5	49 *Auriga*	RD	+ 5.87	−0.13	.49	+1.04	+3.3	+0.4
Oct. 4	28 *Cancri*	DB	+ 4.56	−0.10	.50	−0.84	−2.1	+0.3
Oct. 4	28 *Cancri*	RD	+ 4.56	−0.10	.50	+0.92	+2.6	0.0
Nov. 20	171 *B. Piscium*	DD	+ 8.45	+1.67	.43	−0.91	−1.6	+1.1
Dec. 25	*λ Cancri*	DB	+ 6.40	+0.36	.50	−1.01	−5.0	−1.8
Dec. 25	*λ Cancri*	RD	+ 6.40	+0.36	.50	+1.10	+3.8	+0.3
Dec. 28	*σ Leonis*	RD	+ 5.29	+1.08	.49	+0.83	+0.7	−2.0
1913 Jan. 21	4 *Cancri*	DD	+ 7.17	+0.30	.50	−1.11	−4.9	−1.1
Jan. 21	4 *Cancri*	RB	+ 7.17	+0.30	.50	+0.94	−0.7	(−3.9)
Feb. 15	354 *B. Tauri*	DD	+ 9.79	+0.11	.47	−0.62	−1.8	+0.4
Feb. 25	47 *G. Libra*	DB	+ 7.03	+1.96	.47	−1.00	−4.0	−0.5
Feb. 25	47 *G. Libra*	RD	+ 7.03	+1.96	.47	+0.84	+2.8	−0.1
Mar. 12	47 *Arietis*	DD	+10.78	+0.68	.44	−0.87	−2.1	+1.1
Mar. 17	4 *Cancri*	DD	+10.01	−0.31	.49	−1.07	−6.2	−2.4
Apr. 17	89 *Leonis*	DD	+10.10	+0.36	.50	−1.07	−5.3	−1.4
Apr. 21	47 *G. Libra*	DB	+ 9.07	+1.45	.47	−1.07	−4.5	−0.6
Apr. 21	47 *G. Libra*	RD	+ 9.07	+1.45	.47	+0.97	+3.2	−0.4
Apr. 22	65 *B. Scorpii*	DB	+ 8.68	+1.50	.46	−1.03	−4.2	−0.4
Apr. 22	65 *B. Scorpii*	RD	+ 8.68	+1.50	.46	+0.92	+3.0	−0.4
Apr. 29	182 *B. Aquarii*	DB	+ 8.25	+1.16	.40	−0.72	−6.3	(−3.6)
June 12	*f Virginis*	DD	+10.13	+0.81	.49	−0.90	−4.3	−0.8
June 24	317 *B. Aquarii*	DB	+ 8.08	+0.73	.40	−0.49	−1.4	+0.6
June 24	317 *B. Aquarii*	RD	+ 8.08	+0.73	.40	+0.62	+2.9	+0.4
July 13	*π Scorpii*	DD	+10.12	+1.44	.47	−1.00	−2.6	+1.4
July 21	*h Aquarii*	RD	+ 8.04	+0.41	.40	+0.72	+2.9	−0.1
Aug. 8	17 *G. Libra*	DD	+10.18	+1.70	.48	−1.01	−3.3	+0.9
Sept. 21	4 *Cancri*	DB	+ 7.59	−0.30	.48	−0.74	−2.8	+0.5
Sept. 21	4 *Cancri*	RD	+ 7.59	−0.30	.48	+0.80	+2.2	−1.3
Oct. 20	49 *Auriga*	DB	+ 7.46	−0.16	.46	−0.94	−4.6	−0.3
Nov. 17	134 *B. Geminorum*	DB	+ 7.15	+0.05	.46	−0.94	−5.8	−1.4
Nov. 17	134 *B. Geminorum*	RD	+ 7.15	+0.05	+.46	+1.01	+4.4	−0.2

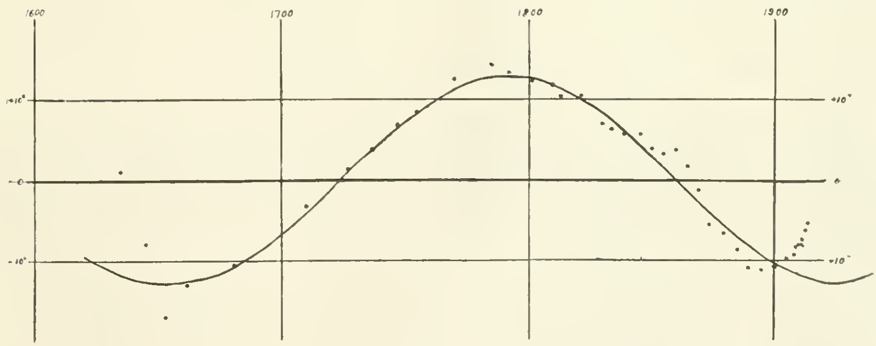
In Table I the columns, "Tabular Correction to HANSEN–NEWCOMB" require explanation. The quantities in these columns have been computed from manuscript tables in the office of the *American Ephemeris*, which purport to include all known sensible corrections to the HANSEN–NEWCOMB position of the Moon given in the *American Ephemeris* since 1883. These corrections comprise the following classes: planetary and solar terms computed by E. W. BROWN; "ellipticity" terms due to G. W. HILL; corrections to the lunar elements, and empirical long period inequality by SIMON NEWCOMB (Ast. Papers, *A. E.*, Vol. 9); and lastly, short period empirical terms computed by the writer (*Monthly Notices*, Vol. 72). Altogether eighty-six periodic corrections to the longitude have been collected and tabulated. These corrections have been applied to the *American Ephemeris* positions of the Moon before computing *D*. The importance of these corrections may be judged by comparing the corrections to the HANSEN–NEWCOMB longitude, $\delta\lambda$, for the dates 1909, July 10 and July 27. In the interval of seventeen days, the correction is seen to have changed from $+0''.59$ to $+9''.30$.

Table II shows the result of the solution of the equations for δM, all the data for which are to be found in Table I. All reappearances at the bright limb were omitted, on account of their systematic uncertainty. The yearly mean value is called M_m. The quantities M_1 are obtained by subducting from M_m the short period empirical terms mentioned above. They are therefore the "minor residuals" of NEWCOMB, or the corrections to the sum, pure theory plus NEWCOMB's long period empirical term. The quantities M_2 are obtained by subducting from M_m *all* empirical terms, and are therefore the corrections to the purely theoretical mean longitude of the Moon.

TABLE II.

Epoch	No. Occs.	M_m	M_1	M_2
		$''$	$''$	$''$
1908.6	5	+0.67	+2.8	−8.2
1909.5	10	+0.81	+3.6	−8.0
1910.8	8	+0.75	+3.7	−8.1
1911.3	5	+1.48	+4.5	−7.4
1912.6	14	+2.87	+5.8	−6.2
1913.4	22	+4.04	+6.9	−5.2

MOON'S OBSERVED LONGITUDE COMPARED WITH THEORY, 1621–1913.



The plate brings down to 1913 NEWCOMB's comparison between the theoretical and observed values of the Moon's mean longitude. The central horizontal line represents pure theory, the dots observation, and the curve, NEWCOMB's long period empirical term. The course of the dots, or observations since 1910, point to the probability of NEWCOMB's long period empirical term breaking down, in the sense of not representing the Moon's longitude save for minor fluctuations. The Moon now appears to be at an epoch in its history in which it is hazardous to attempt to predict its position even two years in advance. Being less amenable to theory and mathematical formulation than has hitherto been supposed, light is shed on the discordant results obtained in the discussion of ancient and mediæval eclipses.

The mean longitude of the Moon is subject to large fluctuations requiring many years to run their course. Are there fluctuations of short period of the order of a lunar month? As the occultations in Table I have been compared with a supposedly perfect theory, the magnitude of the residuals r furnishes a criterion of the existence of fluctuations of this class. These residuals are practically the outstanding errors of δM, assuming that the true value of δM follows the smooth curve obtained from the values M_m of Table II. Assigning a probable error of $0''.25$ to this curve, we find from r that a single occultation gives the Moon's longitude with an accuracy the probable error of which is $0''.78$. The following classes of error are present: errors of observation, errors of star places, errors of the lunar tables, irregularities of the Moon's limb, and lastly the assumed possible short period fluctuations in the Moon's longitude. While this last error, or the one in question, is present in the above deduced probable error, it is not present in a value of the probable error deduced from a number of occultations on the same night. A comparison of these two probable errors will show if such short period fluctuations are present. There is sufficient material in NEWCOMB's Researches, *A.P.*, Vol. 9, Chapter 9, for the determination of the second or restricted probable error. I have computed its value from the occultation there given for three nights on each of which a large number of occultations of bright stars were observed. The mean result was $\pm 0''.92$; with an uncertainty of perhaps $0''.1$. This mean value being of the same order of magnitude as the unrestricted value, $0''.78$, it is clear that there can be no short period irregularities in the Moon's longitude of any appreciable magnitude.

As already remarked, it is doubtful if the Moon's longitude can at the present time be given for even a short period in advance. However, it may be useful to estimate the probable correction to the Moon's longitude at the epoch of the total solar eclipse of August 21, 1914. From the curve described by the values M_m, Table II, it is estimated that the probable correction to the mean longitude at the date of the eclipse is $+6''.5$, or, reduced to right ascension after conversion to true longitude, $+0^s.45$.

The results of transit circle observations of the Sun at Greenwich (*Greenwich Observations*) indicate that its position given by NEWCOMB's Tables needs a sensible correction. According to these observations, the mean yearly correction to NEWCOMB's right ascension varies from zero in 1903 to $+0^s.064$ in 1911. Assuming a further proportionate increase, the correction at the time of the eclipse would be $+0^s.09$.

Combining the above corrections to the lunar and solar right ascensions, the corrections to the time of totality given by the ephemerides become:

Correction to *Nautical Almanac* time....... $-27^s.5$
Correction to *Conn. des Temps* time....... -26.5
Correction to *American Eph.* time.......... -10.5

March 10, 1914.

THE PERIOD OF *o PERSEI*.

By FRANK C. JORDAN.

In the determination of the orbit of this star in *Publications of the Allegheny Observatory*, Volume II, page 63, I have given the period as 4.41916 days, and stated that it is probably correct within 0.0001 days. This result was derived from a comparison of our velocity values with those of VOGEL obtained about six years before. As the star has recently been discovered to be a light variable, possibly of the β *Lyræ* type, it seemed well to examine again the matter of its period. Accordingly twelve additional plates were obtained here with the MELLON spectrograph, data for which are given in the subjoined table:

Plate	Date, G.M.T.	J.D.	V.	Phase	Residuals
	h m		km.	days	days
6059	1913, Oct. 31 18 40	2420072.778	+ 4.88	3.226	0.00
6082	Nov. 14 15 58	86.665	+ 79.29	3.854	+0.12
6093	20 16 35	92.692	+ 25.93	1.043	−0.01
6103	22 17 17	94.720	− 10.29	3.072	−0.04
6115	Dec. 9 14 43	111.613	− 95.83	2.289	

Plate	Date, G.M.T.	J.D.	V.	Phase	Residuals
	h m		km.	days	days
6121	1913, Dec. 11 17 26	2420113.726	+120.76	4.402	
6126	12 15 48	114.658	+ 38.88	0.915	−0.07
6132	13 15 27	115.644	− 84.66	1.900	−0.01
6133	14 17 37	116.734	− 28.55	2.990	−0.01
6140	16 15 10	118.632	+ 88.64	0.469	−0.15
6146	19 16 50	121.702	+ 35.95	3.539	+0.12
6151	27 12 17	129.512	− 87.72	2.511	+0.04

These phases were computed with the formula J. D. 2418217.924 + 4.41916 E, in accordance with the elements given in the paper above mentioned.

The velocities were now plotted on the original computed velocity curve, and the departure from that curve estimated in fractions of a day as given in the column of residuals. The fact that these residuals balance so exactly shows that the original curve and period satisfy the recent observations very closely, and no change in period is required. The question then remaining is that of the probable error of T, upon which will depend the probable error of the period. The epoch of the latest series of observations is T = J.D. 2420096.067 ± 0.016 days. The probable error is computed from the given residuals. Vogel's series (of eighteen plates) may be assumed to give T with at least equal accuracy; hence the probable error of the difference of the two sets is ±0.023 days. Since about 910 periods have elapsed between the two series, the probable error of the period is therefore ±0.000025 days or ±2.2 sec.

Allegheny Observatory, April 16, 1914.

ERRORS IN THE RIGHT ASCENSIONS OF NEWCOMB'S CATALOGUE,

By W. S. Eichelberger and H. R. Morgan.

[Communicated by Captain J. A. Hoogewerff, U.S.N., Superintendent U.S. Naval Observatory.]

The clock list used in reducing the observations made with the 9-inch transit circle of the U. S. Naval Observatory 1903-11, consists of 277 stars selected from Newcomb's *Fundamental Catalogue*. These stars extend from declination −20° to +30°, are fairly uniformly distributed in right ascension, and are more thickly grouped along the zodiac than elsewhere, so that the mean declination at the various hours of right ascension varies slightly with the right ascension.

The standard clock was a self-winding Riefler, with a nickel-steel pendulum and enclosed in an air-tight glass case in which the pressure was maintained quite constant. It was kept in a clock vault of nearly constant temperature, due to artificial temperature control.

All stars brighter than magnitude 6.1 were observed through wire gauze screens, reducing the magnitude 3, 5, or 8 magnitudes, so that all stars were observed as of magnitude between 6 and 9. The object of this was two-fold; first, to eliminate to a large extent the effect of magnitude equation, and secondly, to diminish the accidental error of observation of the brighter stars. The average magnitude of observation of the 277 clock stars was 8.0, and as the deviation from this average was in only a few cases more than one magnitude, no correction has been applied to the observations for the magnitude equation of the several observers, though these equations were determined.

To eliminate from the clock corrections the effect of the magnitude equation of the fundamental catalogue, the ephemeris right ascensions were increased, in the first reduction, by $0^s.02$, $0^s.03$, or $0^s.05$, according as the magnitude of the star observed was decreased by the use of screens, 3, 5, or 8 magnitudes.

All observations were made at night, and not extended far enough into the twilight for daylight to affect the character of the artificial field illumination.

A fixed reticule was used, the transits being recorded on a chronograph by the use of an electric key.

The azimuth of the instrument was given by readings on a north and a south mark, the collimation was the mean of that given by opposing collimators and by the marks, and the level was determined by nadir observations over a mercury basin.

The preliminary azimuths of the marks were determined from groups of observations, half above and half below the pole, of close circumpolar stars whose daily positions are given in the nautical almanacs. The final azimuths were made to depend upon the fundamental positions of these stars resulting from observations, 1908-11, made especially for this purpose. In the determination of the final azimuths there were applied to the transits of the azumith stars corrections for the personal equation of the observers, determined by the use of the personal equation machine belonging to the 9-inch transit circle. The finally adopted fundamental positions of the azimuth stars give the following corrections to Newcomb's right ascensions, each resulting from about fifty separate observations.

Table I.

	From Eye and Ear Observations.						From Chronograph Observations.		
	α	$\Delta\alpha$		α	$\Delta\alpha$			α	$\Delta\alpha$
	h	s		h	s			h	s
λ Urs. Min.	19.1	−0.22	51 H Cephei	7.0	+0.57		ε Urs. Min.	16.9	−0.01
4 B Urs. Min.	8.2	+0.46	39 H Cephei	23.5	+1.38		76 Draconis	20.8	+0.02
α Urs. Min.	1.5	+1.24	δ Urs. Min.	18.0	−0.02		1h Draconis	9.4	−0.01
6 B Urs. Min.	12.2	+0.45	43 H Cephei	1.0	+0.24				
57 B Urs. Min.	15.1	−0.59	151 H' Cephei	4.2	+0.26				

When there were two or more observers on the same night, their relative personal equations were determined from the clock star observations; and from such results the relative equations of the observers were adopted for each observing year.

The ephemeris places having been corrected for the screen correction, each clock correction was reduced to the standard observer by the application of the relative personal equation, and the mean of all the corrections on a given night was then taken as the adopted clock correction from that night's work, the epoch being the mean of the times of transit of the separate clock stars.

The definitive clock rates were computed from rate formulæ obtained by least square solutions of the individual rates obtained from the adopted corrections just mentioned. Into the equation of condition were introduced terms allowing for a variation of the rate with the time, with the temperature, and with the barometric pressure. The separate solutions extended over periods varying from a few weeks to six months. The clock rate was found to vary principally with the time, but often terms depending upon the small barometric and thermometric changes were appreciable. The performance of the clock has been very satisfactory.

As the clock list contained stars extending from declination −20° to +30°, an examination of the differences in the clock corrections as given by stars at different declinations was made as follows:

On each night when there were observations of four or more clock stars between the declinations 0° and +10°, the mean of the clock corrections for these four or more stars was taken and the correction to reduce each observed clock correction to this mean was found. This was done for 152 nights, using 2,373 observations. All these differences were then grouped according to declination and clamp, and the means by groups are given in the following table:

Table II.

Clamp West				Clamp East				West–East		
No. Obs.	Mean Decl.	Results I	Results II	No. Obs.	Mean Decl.	Results I	Results II	Results I	Results II	Results III
	°	s	s		°	s	s	s	s	s
82	−15.3	+0.008	−0.004	189	−14.7	+0.039	+0.015	−0.031	−0.019	−0.005
83	− 7.1	+0.008	+0.002	191	− 7.3	+0.015	+0.002	−0.007	0.000	−0.002
82	+ 0.3	+0.010	+0.008	198	− 0.3	+0.011	+0.001	−0.001	+0.007	0.000
83	+ 3.4	+0.002	0.000	124	+ 3.5	+0.004	+0.001	−0.002	−0.001	0.000
82	+ 5.5	−0.001	0.000	130	+ 5.5	−0.003	0.000	+0.002	0.000	+0.002
82	+ 7.1	−0.002	0.000	130	+ 7.0	−0.002	+0.001	0.000	−0.001	−0.002
83	+ 8.8	−0.002	−0.003	130	+ 9.0	+0.002	0.000	−0.004	−0.003	0.000
82	+12.0	−0.007	−0.001	130	+12.5	−0.004	−0.002	+0.003	+0.001	+0.001
83	+19.5	−0.004	−0.002	130	+19.0	−0.006	−0.007	+0.002	+0.005	+0.002
82	+26.4	+0.002	+0.007	197	+26.0	−0.025	−0.010	+0.027	+0.017	+0.004

In Results I, the clock corrections were obtained by using Newcomb's right ascensions.

In Results II, they were obtained by including the definitive azimuths and the preliminary corrections to Newcomb from the work under discussion. If the quantities were reformed using the final positions of the clock stars, they would be changed, at the most, only a very few thousandths of a second.

In Results III, the number of stars is increased about 50%, the number of observations 3 or 4 fold, and the final positions of the stars resulting from this work are used.

The preliminary clock corrections were affected not

only by this variation with declination but also by the periodic errors in Newcomb's right ascensions varying with the right ascension, and the accidental errors of his individual positions. A definitive clock system was derived from the observations themselves, in which was eliminated almost, if not quite entirely, these three troubles. That the new system eliminated the variation with declination has just been shown. The method of forming this new system was as follows:

Using Newcomb's right ascensions of the clock stars, corrected for the screen correction, and the preliminary values for the azimuths of the marks, that portion of the seven years' work was reduced in which observations were made extending over 7 hours or more on a given night and in which, in addition, 10 or more clock stars were observed on that night. In all, 410 nights were found on which these conditions were satisfied, the average number of clock stars being between 17 and 18 per night, and the average number of hours observing being 10 per night.

Collecting the results of this work, a system of corrections was found to Newcomb's positions of the 277 clock stars, which corrected positions would be comparatively free from the periodic errors in the ephemeris places and but little affected by the errors in the individual places. The average number of observations per star was 26, about equally divided as to clamps.

As stated above, the clock corrections, redetermined by using the results just obtained, still showed a variation with the declination (Results II, Table I), and in an effort still further to reduce this variation a redetermination of the positions of the clock stars was made.

The observations on these same 410 nights were used for the redetermination. Each transit was corrected for the difference in the azimuth as derived from the preliminary and final positions of the marks, each ephemeris place was corrected by the results of the first reduction, and the clock corrections were recomputed. The clock rates were not changed. The right ascensions were corrected for these changes, and new corrections to Newcomb's places were found for each star. The average difference between these corrections and those obtained before, without regard to sign, is $0^s.002$. The preliminary clock system corrected in accordance with this re-reduction was adopted as the definitive clock system.

All the clock corrections for the seven years' work have been recomputed, using these definitive positions of the clock stars, and the corrections for final azimuth; and all observed right ascensions have been corrected to agree with these definitive clock corrections, and the corrections for final azimuth. The resulting system of right ascensions, therefore, will be independent of any other system. The mean right ascensions from this final reduction have not yet been formed.

The individual corrections to Newcomb in the definitive clock system range from $+0^s.11$ to $-0^s.05$. The entire list of these 277 corrections is not given here, but in Table III, the means are shown for every 10° in declination and every 3 hours in right ascension. There is an average of 7 stars in each mean. The right hand column gives the mean correction for each 10° in declination, giving equal weight to each 3 hours of right ascension. The variation with declination is pronounced. The bottom row gives the means for each 3 hours of right ascension, giving equal weight to each 10° of declination, and these means are, therefore, fairly independent of the declination. The average magnitude is the same for each mean.

Table IV was formed by subtracting from the quantities of each row of Table III the mean value of these quantities found in the last column, and shows more clearly the periodic variation of the right ascension correction with the right ascension. Using the means in the next to the last row of Table IV, an analytic expression was obtained giving the periodic variation of this right ascension correction. This expression is given immediately below the table. The values computed by means of the formula are given in the last row of the table. The mean epoch is about 1907.0. The last column of Table IV shows that the variation with right ascension is nearly the same for the five declination groups.

Table III.

Systematic Corrections to the Right Ascensions of Newcomb's *Fundamental Catalogue*.

α / δ	hrs. 0, 1, 2,	hrs. 3, 4, 5,	hrs. 6, 7, 8,	hrs. 9, 10, 11,	hrs. 12, 13, 14,	hrs. 15, 16, 17,	hrs. 18, 19, 20,	hrs. 21, 22, 23,	Mean
° °	s	s	s	s	s	s	s	s	s
+30+20	+0.015	+0.005	−0.001	+0.016	+0.007	+0.013	+0.028	+0.008	+0.011
+20+10	+0.040	+0.019	+0.006	+0.020	+0.018	+0.016	+0.039	+0.031	+0.024
+10+ 0	+0.025	+0.028	+0.004	+0.016	+0.018	+0.038	+0.036	+0.038	+0.026
− 0−10	+0.034	+0.027	+0.010	+0.020	+0.032	+0.044	+0.034	+0.042	+0.030
−10−20	+0.048	+0.054	+0.026	+0.040	+0.047	+0.054	+0.051	+0.061	+0.048
Mean	+0.032	+0.027	+0.009	+0.022	+0.024	+0.033	+0.038	+0.036	+0.028

TABLE IV.

δ \ α	$1^h.5$	$4^h.5$	$7^h.5$	$10^h.5$	$13^h.5$	$16^h.5$	$19^h.5$	$22^h.5$	Range
	s	s	s	s	s	s	s	s	s
+25	−0.004	−0.006	−0.012	+0.005	−0.004	+0.002	+0.017	−0.003	0.029
+15	+0.016	−0.005	−0.018	−0.004	−0.006	−0.008	+0.015	+0.007	0.034
+ 5	−0.001	+0.002	−0.022	−0.010	−0.008	+0.012	+0.010	+0.012	0.034
− 5	+0.004	−0.003	−0.020	−0.010	+0.002	+0.014	+0.004	+0.012	0.034
−15	0.000	+0.006	−0.022	−0.008	−0.001	+0.006	+0.003	+0.013	0.035
Mean	+0.005	−0.001	−0.019	−0.005	−0.003	+0.005	+0.010	+0.008	
Formula	+0.007	−0.004	−0.013	−0.010	−0.002	+0.004	+0.008	+0.010	

$$\Delta\alpha = -0^s.008 \sin\alpha + 0^s.008 \cos\alpha + 0^s.002 \sin 2\alpha + 0^s.002 \cos 2\alpha.$$

Another series of observations made from August, 1909, to April, 1911, on quite a different plan, as far as they give any information on this subject, confirm the results of the series just discussed. Two groups of six stars each, about 12 hours apart, the one in 6 hours right ascension and the other in 18 hours, were observed continuously during the period named, except the group that transited within two or three hours of the Sun. These groups transit at 6 A.M. and 6 P.M., at the equinoxes; and from 729 observations at four sucessive equinoxes, the mean correction to NEWCOMB was found to be $0^s.029$ greater for the 18-hour group, than for the 6-hour group. From the individual corrections for these 12 stars, as found in the clock list of 277 stars, the mean correction for the 18-hour group is $0^s.026$ greater than that for the 6-hour group.

The subject of the periodic error in the right ascensions of the fundamental catalogue has been treated by several writers recently. In *Astronomische Nachrichten*, No. 4668, from a discussion of Cape, Greenwich, Pulkowa and Odessa observations, COHN finds the periodic correction to AUWERS right ascension to be

$$\Delta\alpha = -0^s.007 \sin\alpha + 0^s.007 \cos\alpha + 0^s.001 \sin 2\alpha + 0^s.006 \cos 2\alpha.$$

And in the *Year Book, 1912,* of the Carnegie Institution of Washington, p. 168, from the San Luis observations, BOSS gives the periodic correction to his *Preliminary General Catalogue* as

$$\Delta\alpha = -0^s.008 \sin\alpha + 0^s.009 \cos\alpha.$$

OBITUARY NOTE.

The *Astronomical Journal* announces with sorrow the death of an Associate Editor, GEORGE WILLIAM HILL. Doctor HILL died at his home in West Nyack, N. Y., on April 16, 1914. At the request of the Editor, Dr. R. S. WOODWARD has prepared a sketch of HILL's life which arrived too late to be incorporated in the present issue. It will appear in the next number.

NOTES.

CORRECTION to observation of Comet 1911 (*f*) (OPIENISSET), published in *Astronomical Journal* No. 653:

1912	Gr.M.T.	$\Delta\alpha$	$\Delta\delta$	App. α	App. δ	log $p\Delta$ α	log $p\Delta$ δ	Red. to App. Pl. α	Red. to App. Pl. δ
Febr. 16	19 40 18	-0^m 51.04	$+6'$ 43''.1	14 53 27.58	$-63°$ 27' 44''.8	9.700*n*	0.599	$-0^s.19$	$+7''.8$

COMPARISON STAR: Cordoba *Gen. Cat.*, No. 20.297 1912.0 α 14^h 54^m $18^s.81$ δ $-63°$ 34' 35''.7

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ACTING EDITOR, BENJAMIN BOSS, ALBANY, N. Y.; ASSOCIATE EDITOR. PROF. ERNEST W. BROWN, OF YALE UNIVERSITY. PUBLISHED BY THE DUDLEY OBSERVATORY, ALBANY, N. Y., U.S.A., TO WHICH ALL COMMUNICATIONS SHOULD BE ADDRESSED. PRICE, $5.00 THE VOLUME. PRESS OF THOS. P. NICHOLS & SON CO., LYNN, MASS. *Closed, May* 16, 1914.

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GEORGE WILLIAM HILL.

George William Hill, long a frequent contributor to this *Journal* and more recently one of its Associate Editors, died at his home in West Nyack, N. Y., on April 16, 1914, at the age of seventy-six years, one month and thirteen days. He was of English-Huguenot descent and was born in New York City March 3, 1838. His early education appears to have been without noteworthy incident until he entered Rutgers College, where he attained the degree of A. B. in 1859, and where he had the good fortune to be introduced while yet an undergraduate to the *Mécanique Céleste* of Laplace. This introduction was furnished by Dr. Theodore Strong (1790–1869), then professor of mathematics and natural philosophy in Rutgers College, and one of the small number of Americans devoted to mathematico-physical science at that time. Strong had a good library in this science to which his pupil was given free access, and Hill often referred in terms of grateful appreciation to this circumstance as one of the determining factors in his remarkable career of research in dynamical astronomy. That he was a student worthy of special privileges is evident from his earliest investigations. He began independent research, indeed, before attaining the baccalaureate degree; while shortly after he published (as a prize essay, in Runkle's *Mathematical Monthly*, 1861) a very noteworthy paper in which he attempts to extend Laplace's investigation on the figure of the Earth. This paper is still worthy of careful reading; and it is especially instructive in showing that Hill had already, at the age of twenty-one years, acquired that mastery of analytical resources and that facility in exposition which enabled him to become the leading contributor to the advances in dynamical astronomy during the following half century.*

Happily for Hill and for astronomical science there was offered him soon after leaving college a position in the *Nautical Almanac* Office, then maintained at Cambridge, Mass., but later (1867) transferred to Washington and brought under the more immediate jurisdiction of the Navy Department. This led him into close association with the ablest of his American contemporaries in astronomical research, and afforded the amplest of experience in numerical computations in which he became an expert of the high order essential to the stupendous calculations in planetary theory he later undertook in collaboration with Simon Newcomb. It is a remarkable and highly significant circumstance to which educators and statesmen should give heed, that out of this *Nautical Almanac* Office, intended primarily as a practical aid to navigation, there should have come from these two men chiefly not only many important contributions to practical astronomy but most of the more important contributions to theoretical astronomy of the second half of the nineteenth century. Early during his connection with the *Nautical Almanac* Office, Hill prepared for the needs of computers, navigators and geographers a manual of formulas and tables for the derivation of stellar positions; and later on, in anticipation of the transits of *Venus* of 1874 and 1882 he prepared a similar manual for the American Transit of *Venus* Commission. He was thus ever ready and effective in supplying the wants of the more obviously utilitarian applications of astronomy. But while he never contemned such applications and spoke often in admiration

* In some brief autobiographical notes he gives the following glimpse of his college career:

"Having shown some aptitude for mathematics it was decided to send me to college; and, in October, 1855, I took up residence at Rutgers College, New Brunswick, N. J. Here I found Dr. Theodore Strong, professor of mathematics, who was a friend of Dr. Nathaniel Bowditch, the translator of Laplace's *Mécanique Céleste*. I remember seeing in Dr. Strong's library the presentation copy of this work. Under his guidance, I read such books as Lacroix, *Traité du Calcul Différential et Intégral;* Poisson, *Traité de Mécanique;* Pontécoulant, *Théorie Analytique du Systime du Monde;* Laplace, *Mécanique Céleste;* Lagrange, *Mécanique Analytique;* Legendre, *Fonctions Elliptiques*. My professor was an old-fashioned man; he liked to go back to Leonard Euler for all his theorems; as he said 'Euler is our Great Master.' He scarcely had a book in his library published later than 1840."

of the exquisite perfection of the observational astronomy of his day, it was plain that his deeper interests were attached to the more recondite field of gravitational astronomy. To this field nearly all his leisure during a term of thirty years of service as assistant in the *Nautical Almanac* Office was sedulously devoted, and the advances he made in lunar and planetary theory won him, by common assent of his peers, a position of pre-eminence in this domain of astronomical science.

Naturally the large aggregate of fundamental work accomplished by HILL made him a voluminous writer. He published extensively in the current journals and in the professional papers of the *Nautical Almanac* Office. He early adopted the practice of printing promptly whatever had passed the tests of his critical judgment; and his exposition is everywhere marked by a clearness and a conciseness as characteristic as the originality of his concepts and the adequacy of his methods. His "Collected Mathematical Works" have been published recently (1905–1907) by the Carnegie Institution of Washington in four quarto volumes.* The first of these volumes contains an introductory biographical and historico-critical chapter of appreciation in French by HILL's eminent contemporary and friend, POINCARÉ.

Unfortunately for most of his contemporaries, but perhaps fortunately for him, HILL was too little known and hence too little appreciated, especially by his fellow countrymen. He was so absorbingly preoccupied with his researches that he had scant leisure for association even with his more intimate friends. He held himself mostly apart from the ordinary affairs and amenities of society and consecrated his energies to the less obvious realm of celestial mechanics. But this was not because he lacked the social instincts common to mankind. To those who knew him well he presented a singularly attractive personality. Although his innate modesty obscured his superior talents in a large or in a mixed company, he was a delightful companion for a long walk and a very stimulating host to, or guest of, his chosen friends individually. His transparent sincerity, his steadfast but unostentatious devotion to elevated ideals and the effective simplicity of his life will be long remembered as among the finer qualities of the man HILL by those who had the good fortune to come into closer assocaition with him.

After retiring from the *Nautical Almanac* Office in 1892, HILL devoted the remaining twenty-two years of his life almost exclusively to his favorite researches. During this interval, however, he gave no inconsiderable time and effort to coöperation in developing the American Mathematical Society, of which he was President in 1894-6; while he served as lecturer on celestial mechanics in Columbia University during the academic years 1898 to 1901. His lectures on this subject, read before a necessarily narrowly limited group of advanced students, were in a highly finished form. Each auditor was furnished with a complete hectograph copy made from manuscript carefully written out by HILL himself. He did not think it fitting, however, to include these lectures as part of his "Collected Works," referred to above. His productive activity continued throughout these later years with little abatement, except for the side issues just mentioned, and his mental faculties appear to have retained their characteristic clearness and serenity to the end. During the past year, in fact, he has published in this *Journal* several papers wherein one finds the same precision, elegance and penetration with which he has so long charmed and instructed his readers.

Although almost unknown by the learned world at large, and little known personally by his contemporaries in astronomical science, the merits of HILL's published work were widely recognized by his peers and by learned societies and institutions at home and abroad. He was a recipient of many honors. These were not unappreciated, they were in fact warmly esteemed by him, but in all of them to a peculiar degree he figures as the altruistic man of science rather than as an individual of distinction. His attitude was that of a man whose talents were applied in freest and fullest measure to the progress of his fellow men and he asked of them only opportunity to labor unceasingly.

HILL never married. His mode of life, whether in his residence in the city of Washington or in his home at West Nyack, N. Y., was one of Spartan but dignified simplicity. His labors were broken only by long walks in the vicinity of Washington, whose environs he well knew, by occasional visits to his more intimate friends and by less frequent vacation excursions to distant parts of the world. He had a strong liking for geographical exploration and he made two trips for this purpose into Canada, one into the Hudson Bay region and one into the Canadian Northwest. It was during his journey through the latter terriotory that he worked out his famous solution of the problem involving an infinite determinant, a solution "*aussi originale que hardi*," as remarked by POINCARÉ. He was thus able to maintain bodily vigor and to meet the inexorable requirements of the arduous science to which he was alone wedded; and he was thus able to pass on to his successors extensive additions to that sort of knowledge whose certainty and permanence are comparable with the continuity of the seasons and the constancy of the stars.

* Papers since published by HILL will suffice to make an additional volume.

R. S. WOODWARD

DEPENDENCE OF SOLAR MOTION UPON SPECTRAL TYPE,

By BENJAMIN BOSS.

Since no discussion of stellar motions is free from the effects of solar motion, it seems regrettable that the determination of this fundamental element of motion should be liable to so great a degree of uncertainty. When, as will be shown later on, the same material will yield different results according to the method of grouping, it is little to be wondered at that different methods and different data should produce results greatly at variance with one another. But the wide range over which solar motion determinations extend might lead one to suspect that part of the effect was due to a real cause.

In *A. J.* 649–650, p. 15, a preliminary note calls attention to an apparent dependence of the direction and amount of solar motion upon spectral type. The table demonstrating the effect is repeated below.

Table I.

Type	A (°)	D (°)	V (km)
K	275.4	+40.3	−21.2
M	273.6	+38.8	−22.6
A	270.0	+28.3	−16.8
F	265.9	+28.7	−15.8

The positions of the apex were taken from *A. J.* 623-624 where Professor Lewis Boss made the separate determinations by type, and the velocity of the solar motion, as given in the last column, is taken from Professor Campbell's paper in *L. O. B.*, 196.

The two groups composed respectively of late and early type stars stand out distinctively both as regards direction and velocity of solar motion. Considerable weight should be assigned to the results since they were derived by similar methods from homogeneous material. The positions of the solar apex lie approximately along a parallel of about +22° galactic latitude. In *A. J.* 655 Professor Comstock classified according to the mean magnitude of the stars employed, the results for the solar apex as determined by various authorities who had used the same method but non-homogeneous data. He arrived at the conclusion that the position of the apex of solar motion undergoes a progressive increase in galactic longitude with diminution in the mean magnitude of the stars employed, the latitude remaining the same.

The general similarity between Professor Comstock's results and mine suggested that they might both be assigned to the same cause.

In order to attain results of any weight it is very necessary to use absolutely homogeneous material and the same method. Consequently the stars of the *Preliminary General Catalogue* were divided into three classifications according to magnitudes, the limits being 0 − 5.2; 5.3 − 6.0; 6.1 and fainter. The proper motions of each classification were grouped according to equatorial areas, and solved by the methods described in *A. J.* 612 and 614. The normal equations in right ascension and declination have been combined to give the following normal equations for each class.

$0^M - 5^M.2$

$$\begin{array}{rrrr} +265.34\,X & +\ 2.22\,Y & -\ 12.06\,Z & = -\ 73.26 \\ +\ 2.22 & +240.37 & -\ 1.41 & = -834.47 \\ -\ 12.06 & -\ 1.41 & +245.42 & = +609.21 \end{array}$$

$5^M.3 - 6^M.0$

$$\begin{array}{rrrr} +318.36\,X & +\ 4.19\,Y & -\ 10.16\,Z & = -\ 26.86 \\ +\ 4.19 & +294.64 & -\ 1.22 & = -904.10 \\ -\ 10.16 & -\ 1.22 & +290.36 & = +624.59 \end{array}$$

$6^M.1$ and Fainter

$$\begin{array}{rrrr} +276.33\,X & +\ 4.32\,Y & +\ 0.85\,Z & = +\ 40.18 \\ +\ 4.32 & +284.22 & -\ 0.62 & = -824.86 \\ +\ .85 & -\ 0.62 & +269.57 & = +545.59 \end{array}$$

The results from the solution of these equations are given in Table II.

Table II.

Limits of Mag.	Mean Mag.	No. **	X	Y	Z	A	D	M	λ	β
M M	M		″	″	″	°	°	″	°	°
0−5.2	4.4	1442	−.14	−3.39	+2.46	267.7	+35.9	4.19	29.1	+25.2
5.3−6.0	5.8	2067	+.02	−3.06	+2.14	270.4	+35.0	3.73	28.8	+22.8
6.1−	6.7	1864	+.18	−2.90	+2.02	273.6	+34.8	3.54	29.5	+20.3

It becomes evident from the last two columns of Table II that there is no appreciable change in the galactic longitude of the solar apex with decreasing magnitude, the recorded change, on the contrary, being one of latitude. The probable error of solar apex determinations is such that it would be impossible to say whether the noted change in latitude with decreasing magnitude is real or not. The range of magnitudes is not great but is sufficient to record any appreciable change in the galactic longitude of the solar apex if such an effect were real.

A second solution of the same material was made combining with each north equatorial area its diametrically

opposite south equatorial area. The normal equations are:

$0^{M} - 5^{M}.2$ (Combined Areas)

$$\begin{array}{rrrl} +169.13\,X & +\;\;3.73\,Y & -\;\;1.10\,Z & = -\;\;29.11 \\ +\;\;3.73 & +161.96 & -\;\;1.07 & = -528.43 \\ -\;\;1.10 & -\;\;1.07 & +161.96 & = +386.43 \end{array}$$

$5^{M}.3 - 6^{M}.0$ (Combined Areas)

$$\begin{array}{rrrl} +\;38.86\,X & +\;\;0.05\,Y & -\;\;0.20\,Z & = +\;\;\;1.06 \\ +\;\;0.05 & +\;38.49 & +\;\;0.18 & = -120.82 \\ -\;\;0.20 & +\;\;0.18 & +\;35.61 & = +\;\;75.85 \end{array}$$

$6^{M}.1 -$ (Combined Areas)

$$\begin{array}{rrrl} +173.05\,X & +\;\;2.74\,Y & +\;\;1.18\,Z & = +\;\;24.24 \\ +\;\;2.74 & +178.63 & +\;\;0.93 & = -514.02 \\ +\;\;1.18 & +\;\;0.93 & +167.21 & = +337.29 \end{array}$$

The conditional equations were weighted in all cases excepting $5^{M}.3 - 6^{M}.0$ (Combined Areas) where each area had enough stars to render weighting unnecessary. Weights were employed roughly proportional to the square roots of the number of stars in each area. The results for the combined area solutions are given in Table III.

Table III.

Limits of Mag.	Mean Mag.	No. **	X	Y	Z	A	D	M	λ	β
M M	M		″	″	″		°	″	°	
0 − 5.2	4.4	1442	−.08	−3.24	+2.36	268.5	+36.1	4.02	29.4	+24.7
5.3 − 6.0	5.8	2067	+.04	−3.15	+2.15	270.8	+34.3	3.84	28.2	+22.4
6.1 −	6.7	1864	+.17	−2.89	+2.03	273.4	+35.1	3.54	29.8	+20.6

A comparison of Tables II and III indicates that even identical material may yield slightly different results according to its arrangement. This is again shown by comparing the general solution as derived by Prof. L. Boss in *A. J.* 623-624 with the general solution which has been derived by combining according to galactic coördinates.

Equatorial Areas	= 270°.5	+34°.3
Galactic Areas	= 271 .5	+33 .7

It will likewise be noted that the general solution by equatorial areas differs by about 1° with the means of Tables II and III though the identical stars were employed. This is caused by a difference in the relative weights of the area groupings.

Thus we have strong evidence that the variations in the position of the solar apex have little if any dependence upon the mean magnitude of the stars employed as far as the material treated is capable of determining such a relation. Certainly the effect due to mean magnitude is small as compared with the effect due to type.

The G type stars might seem to be anomalous, for the position of the apex of solar motion as derived from the stars of this type (R.A. 259°.3 Decl. + 12°.3*) is placed at a considerable distance from the other determinations. This case has already been treated in *A.J.* 649-650, where it was shown that the G type stars employed in the determination of the direction of solar motion had a strong preference for motion toward Kapteyn's Vertex II. However, as was also pointed out, the G type stars of centennial proper-motion of over 20″ which were excluded from the solution tend strongly toward Vertex I, so that their inclusion would considerably modify the solar motion determination. Unfortunately, the inclusion of the large proper-motion stars influences the results unduly, to so great an extent that the small proper-motions have little effect on the solution. Consequently, in order to include them some modification had to be made. The ideal conditions would be to reduce the large proper-motions to the scale of the mean parallax of the small proper-motion stars employed, but lacking the required parallaxes the large proper-motion stars were all arbitrarily reduced to 20″ centennial motion. A solution for solar motion was made on this hypothesis placing the apex at R.A. = 277°.4, Decl. = +29°.1. Thus, by including the large proper-motion stars of type G according to the methods delineated, the apex of solar motion has been swung to the opposite side of the galactic parallel +22°. It can be readily seen that by adopting a smaller value of the centennial motion for the large proper-motion stars the position of solar motion would very approximately have been placed near +22° galactic latitude in a position lying between the type A and F results and the type K and M results. Where parallaxes exist for these stars a smaller centennial proper-motion is actually indicated. Thus, it seems to supply the missing link.

It might be expected a priori that the apex of solar motion as derived from the B type stars (A = 274°.4, D = +34°.9*) would be located near the positions as determined from Types A and F. The fact that it does not conform is due most probably to the small peculiar motions of this type, and possibly in part to the peculiar uneven distribution in a limited galactic equatorial zone.

It might be well to state here that peculiar distribution could have no effect upon the solutions from other types since these solutions were based upon a division of the sky into areas, the same divisions being used in the case of each solution. For each areal division the mean of the X, Y, Z coefficients and the mean of the proper-motions was used in forming equations of condition.

* *A.J.* 623-624.

It remains to be seen what evidence radial velocities have to give on the subject. Unfortunately, the material in this case is, comparatively speaking, very limited, so that the probable error of a determination of the direction of solar motion according to division by type is very large. However, a tendency such as is evidenced by the proper-motion solutions is sufficiently strong to lead us to hope for some confirmation from radial velocity solutions for solar motion. Therefore the radial velocities as published by Professor CAMPBELL in *L. O. B.* 195, 211 and 229 were divided according to the same classifications by type employed in the proper-motion solutions. The formula used was —

$$\sin\delta\,.\,X + \cos\delta\,\cos\alpha\,.\,Y + \cos\delta\,\sin\alpha\,.\,Z + \rho = 0$$

where $X = V \sin D$; $Y = V \cos D \cos A$; $Z = V \cos D \sin A$; ρ = observed stellar velocity; V = Sun's velocity; A, D = coördinates of apex; α, δ = coördinates of star.

$$V^2 = X^2 + Y^2 + Z^2$$

$$\operatorname{Sin} D = \frac{X}{V} \qquad \tan A = \frac{Z}{Y}$$

The first set of solutions given in Table IV are based upon equations of condition formed from individual stars.

TABLE IV.

Type	No. **	A	D	V	Corr. to Lick
		°	°	km	km
$B_0 - B_5$	179	272.2	+36.8	−23.0	− 2.3
$B_8 - A_4$	228	260.3	+27.5	−18.3	− 1.5
$A_5 - F_9$	248	267.9	+17.8	−17.5	− 1.7
G	147	259.7	+28.0	−26.1	−10.1
G′	138	259.9	+18.9	−17.2	− 1.2
K	439	272.1	+29.7	−23.7	− 2.5
M	80	271.8	+27.8	−25.2	− 2.6
All	1321	268.9	+28.7	−21.6	− 2.1

The first column gives the classification by type. The second column contains the number of stars employed in each solution. Columns three and four give the R.A. and Decl. of the apex of solar motion, and column five the solar velocity. Column six is a comparison of the values for the velocity of solar motion as here obtained with the values published by CAMPBELL in *L. O. B.* 196. As can be seen, the results of Table IV consistently increase the values for solar motion as given by CAMPBELL, undoubtedly due to the fact that more large radial velocities were used than those employed by CAMPBELL.

The row G′ refers to a solution of the G type radial velocities omitting nine of the largest values. The effect is to swing the position of the apex of solar motion nine degrees, and to change the solar velocity derived from that type by nine kilometers. The result obtained from G′ is consistent with the value obtained by CAMPBELL, and is most probably nearer the true value.

In view of the sensitiveness of the solutions only an indication of the results obtained from proper-motions could be hoped for, but that indication exists. In fact, with the exception of the A type ($B_8 - A_4$), the results are almost a reproduction of the proper-motion results, simply shifted by about ten or eleven degrees along the parallel of +22° galactic latitude toward decreasing galactic longitude.

The B type stars are anomalous again, the results from radial velocity placing the apex of solar motion at almost the identical spot where it was located by the treatment of proper-motions. The close agreement of the two methods would seem to indicate that either the small real motions of the B type stars permit a more definite solution of solar motion, their peculiar distribution effects the problem, or there is a strong one drift.

Lest uneven distribution should play too great a part in the problem, the radial velocities were next divided into groups for each type and solved. The results are given in Table V.

TABLE V.

Type	A	D	V	Corr. to Lick
	°	°	km	km
B	278.4	+31.6	−22.9	−2.2
A	267.4	+27.4	−17.9	−1.1
F	263.6	+15.7	−17.7	−1.9
G	263.7	+25.6	−24.3	−8.3
G′	259.5	+ 9.5	−15.5	+0.5
K	269.4	+30.3	−24.7	−3.5

There were too few M type stars to permit a solution by groups.

The G′ row refers to a solution in which the eight largest radial velocities were omitted.

Since the material for individual types is so limited types A and F, and types K and M were combined for both proper-motions and radial velocities and a comparison of these results is given in Table VI.

TABLE VI.

	Proper-Motions				Radial Velocities			
	A	D	λ	β	A	D	λ	β
	°	°	°	°	°	°	°	°
K , M	274.6	+41.2	34.8	+21.7	269.9	+27.9	21.5	+20.9
A , F	268.9	+28.9	22.2	+22.1	261.3	+22.2	12.6	+26.2
Diff.	+5.7	+12.3	+12.6	− 0.4	+8.6	+ 5.7	+ 8.9	− 5.3

While the agreement is not perfect there is a strong similarity between the two sets of values given in Table VI. The velocity of solar motion as derived from K and M type stars is 24.0 kilometers, and that from A and F types 17.7 kilometers, a difference of 6.3 kilometers.

Thus far solar motion has been treated for dependence upon mean magnitude and spectral type. It also seemed advisable to solve according to galactic distribution. Accordingly lunes of galactic longitude were formed forty degrees in width extending from north to south galactic pole. The solutions of these lunes for solar motion are contained in Table VII.

Table VII.

Galactic Lunes	Mean Mag.	No. *	X	Y	Z	A	D	M	λ	β
°	M		″	″	″	°	°	″	°	°
340 – 20	5.9	633	−0.23	−2.31	+1.93	264.4	+39.8	3.02	32.5	+28.8
20 – 60	5.7	560	+0.42	−2.88	+2.51	278.2	+40.7	3.85	36.8	18.9
60 – 100	5.7	610	+0.25	−3.86	+1.95	273.8	+26.7	4.33	20.1	17.3
100 – 140	5.7	623	+0.39	−3.40	+1.86	276.6	+28.6	3.89	24.5	15.7
140 – 180	5.6	592	−0.15	−3.05	+1.89	267.2	+31.7	3.59	24.5	24.4
180 – 220	5.6	558	+0.08	−3.65	+2.28	271.2	+32.0	4.30	26.0	21.3
220 – 260	5.6	565	+0.02	−2.89	+2.60	270.4	+41.9	3.88	36.1	24.9
260 – 300	6.0	540	+0.22	−3.87	+2.04	273.3	+27.4	4.36	22.2	17.9
300 – 340	5.7	568	−0.24	−2.82	+2.04	265.1	+35.5	3.47	28.0	27.2

The first column gives the limits of galactic longitude for each lune, the second the mean magnitude of the stars employed, the third the number of stars, the fourth, fifth and sixth the solution of the normal equations, the seventh the right ascension of the apex, the eighth the declination of the apex, the ninth the mean parallactic centennial motion due to the Sun, the tenth the galactic longitude, and the eleventh the galactic latitude. There is no evidence, whatever, of any systematic tendency in the distribution of the apices of solar motion resulting from the solution in lunes of galactic longitude. A rearrangement according to mean parallactic motion as given in Table VIII might possibly be said to indicate a slight tendency toward decreasing galactic longitude and decreasing galactic latitude of the apex of solar motion with increasing mean parallax. However, the tendency is not strong enough to be convincing when one considers the probable errors to which such solutions are subject.

Table VIII.

M	λ	β
″	°	°
3.02	32.5	+28.8
3.47	28.0	27.2
3.59	24.5	24.4
3.85	36.8	18.9
3.88	36.1	24.9
3.89	24.5	15.7
4.30	26.0	21.3
4.33	20.1	17.3
4.36	22.2	17.9

The material was next divided into twenty degree zones of galactic latitude, and again solved, the results forming Table IX, which is similar to Table VII.

Table IX.

Galactic Zone	Mean Mag.	No. *	X	Y	Z	A	D	M	λ	β
°	M		″	″	″	°	°	″	°	°
+70 to +90	5.9	109	+0.01	−2.91	+2.77	270.2	+43.6	4.02	37.9	+25.5
+50 to +70	5.9	354	+0.42	−3.44	+2.30	277.0	+33.6	4.17	29.4	17.3
+30 to +50	5.9	636	−0.13	−3.28	+2.06	267.6	+32.1	3.87	25.1	24.2
+10 to +30	5.7	933	+0.05	−3.48	+2.19	270.9	+31.6	3.87	28.6	22.4
−10 to +10	5.5	1316	−0.39	−2.91	+1.73	262.3	+30.6	3.41	21.9	28.1
−30 to −10	5.6	1016	−0.03	−3.36	+2.02	269.5	+31.1	3.92	24.6	28.7
−50 to −30	5.8	612	−0.04	−3.54	+2.03	269.4	+29.8	4.08	23.2	22.0
−70 to −50	5.9	106	+0.31	−2.71	+2.10	276.5	+37.7	3.41	33.3	+19.2
−90 to −70	6.0	116	+1.30	−2.18	−0.05	332.2	− 0.7	4.69	29.4	−44.2

With the exception of a possible slight tendency toward similarity between zones symmetrically placed with regard to the galactic equator there seems to be nothing of a systematic character. Again, when as in Table X an arrangement is made according to mean parallax there is no evidence of any systematic arrangement.

But in the last row of Table IX there is a marked deviation from the mean value. This was at first considered

Table X.

M	λ	β
″	°	°
3.41	21.9	+28.1
3.44	33.3	19.2
3.87	25.1	24.2
3.87	28.6	22.4
3.92	24.6	28.7
4.02	37.9	25.5
4.08	23.2	22.0
4.17	29.4	+17.3
4.69	29.4	−44.2

as solely due to the few stars in the region around the south galactic pole and the weakness of their proper-motions. However, it was investigated and will form the subject of a subsequent paper.

Summary.

Insofar as the proper-motions of the *Preliminary General Catalogue* can be relied upon to furnish consistent results, the following deductions are indicated in the foregoing investigation.

There is little if any shift in galactic longitude in the position of the apex of solar motion dependent upon the mean magnitude of the stars employed in its determination though a slight and possibly fictitious shift in galactic latitude is noted.

On the other hand, there seems to be a considerable dependence of the position of the apex of solar motion upon spectral type, probably due to a relative drift between early and late type stars. It manifests itself in a separation in galactic longitude of about 12° between the positions of the solar apex as derived from early and late type stars, the galactic latitude remaining constant at about +22°. The evidence on this point is satisfactorily corroborated by a similar treatment of radial velocities. It also manifests itself in a difference of about six kilometers between the values of solar velocity as determined from early and late type stars.

There is, however, a systematic difference between the position of the apex of solar motion as determined from proper-motions and that determined from radial velocities, apparently of the same order as the difference due to types.

The close agreement of the solar apex as deduced from the B type stars from both proper-motions and radial velocities indicates one of two extremes. Either the small real velocities of these stars fit them especially for a solar motion determination, or there exists among them a one drift which entirely unfits them for such a determination. Until the real cause of the agreement can be established it might be more advisable to exclude the B type stars from the determination of solar motion.

If there is any real effect on solar motion due to the distribution of the stars according to galactic longitude or latitude, the existing material is not capable of revealing it.

In the future, discussions of solar motion should take into account the effect of spectral type, since manifestly a change in the ratio between early and late type stars will cause a shift in the position of the solar apex. It might be advisable for the present to adopt as the position of the solar apex a mean value between early type and late type solutions A = 272°, D = +35° from proper-motions; A = 266°, D = +25° from radial velocities; or for general purposes the position A = 270°, D = +30°.

OBSERVATIONS OF COMET *f* 1913 (*Delavan*).

MADE WITH THE 12-INCH EQUATORIAL AT THE NATIONAL ARGENTINE OBSERVATORY, CORDOBA.

BY A. ESTELLE GLANCY.

1913, 1914 Greenwich M.T.		*	Comp. α	Comp. δ	Δα	Δδ	App. α	App. δ	log p Δ α	log p Δ δ	Red. to App. Pl. α	Red. to App. Pl. δ	Notes.
	h m s				m s	′ ″	h m s	° ′ ″			s	″	
Dec. 18	16 20 44	1	5t	5	+2 10.32	−7 31.2	3 02 20.37	−7 19 14.2	9.536	0.583n	+4.11	+19.3	Clouds
19	17 10 01	1	7t	7	+1 16.02	−1 31.4	3 01 26.06	−7 13 14.5	9.623	0.602n	+4.10	+19.2	Clouds
22	13 14 04	2	10t	10	+1 05.67	−6 16.8	2 59 00.60	−6 55 57.4	7.679	0.559n	+4.07	+18.9	
24	12 34 18	2	10t	10	−0 31.52	+6 22.7	2 57 23.40	−6 43 18.2	8.792n	0.563n	+4.06	+18.6	
Jan. 3	13 14 53	3	10t	10	+1 8.92	+4 43.3	2 50 05.95	−5 31 14.8	9.109	0.584n	+0.99	+ 3.5	
8	13 32 40	4	11	9	− 5.91	+1 31.2	2 47 05.60	−4 51 02.2	9.325	0.599n	+0.93	+ 3.1	
16	13 21 49	5	10	10	+ 8.73	−2 29.4	2 42 57.89	−3 37 40.3	9.417	0.616n	+0.82	+ 2.9	
Feb. 5	13 21 38	6	10t	10	+2 2.29	−1 47.4	2 39 00.02	−0 29 25.2	9.596	0.655n	+0.52	+ 2.6	
9	13 20 27	7	6t	6	+1 12.55	+1 55.2	2 38 47.15	+0 12 56.8	9.616	0.661n	+0.47	+ 2.9	Setting
13	12 40 40	8	10	9	+ 1.86	−1 22.1	2 39 2.00	+0 54 21.6	9.582	0.667n	+0.41	+ 3.0	
Mar. 9	12 27 47	9	10t	10	−1 59.20	+5 39.6	2 45 44.00	+5 15 2.0	9.669	0.681n	+0.14	+ 4.0	

t signifies transits. Observations taken on Dec. 26, 29, Jan. 26 will be published later, after the positions of the comparison-stars have been determined.

Mean Places of the Comparison–Stars for the beginning of the year.

*	α	δ	Authority		*	α	δ	Authority	
	h m s	° ′ ″				h m s	° ′ ″		
1	3 00 05.91	−7 12 02.3	A.G.Wien-Ottakring	696	6	2 36 57.21	−0 27 40.4	A.G. Nicolajew	551
2	2 57 50.86	−6 19 59.5	A.G.Wien-Ottakring	685	7	2 37 34.13	+0 10 58.7	A.G. Nicolajew	554
3	2 18 56.04	−5 36 1.6	A.G.Strassburg	690	8	2 38 59.73	+0 55 40.7	A.G. Nicolajew	561
4	2 47 10.58	−4 52 36.5	A.G.Strassburg	685	9	2 47 43.06	+5 09 18.4	A.G. Albany	801
5	2 42 18.34	−3 35 13.8	A.G.Strassburg	669					

OBSERVATIONS OF COMET *a* 1914 (*KRITZINGER*),

MADE WITH THE 12-INCH EQUATORIAL AT THE NATIONAL ARGENTINE OBSERVATORY, CORDOBA.

By A. ESTELLE GLANCY.

1914 Greenwich M.T.	*	Comp. α δ	Δα	Δδ	App. α	App. δ	log p Δ α	log p Δ δ	Red. to App. Pl. α	Red. to App. Pl. δ	Notes
h m s			m s	′ ″	h m s	° ′ ″			s	″	
Mar. 30 18 57 17	1	7*t* 7	+1 1.91	−2 19.1	16 15 15.77	−8 54 15.2	9.134*n*	0.528*n*	+1.79	−16.8	Clouds. Nucleus. Brightness=10m.5
Apr. 1 17 0 52	2	10*t* 10	−2 55.73	+5 40.4	16 21 16.14	−7 50 49.2	9.551*n*	0.579*n*	+1.81	−16.9	
1 17 41 37	3	10*t* 9	+0 46.23	−3 14.9	16 21 21.68	−7 50 1.1	9.448*n*	0.563*n*	+1.82	−17.0	
2 16 17 54	4	13*t* 12	−1 26.78	+3 27.1	16 24 26.92	−7 16 30.1	9.573*n*	0.590*n*	+1.83	−17.1	
5 20 23 26	5	7*t* 6	+2 28.44	+5 52.0	16 34 18.65	−5 18 56.8	8.767	0.585*n*	+1.89	−17.6	Clouds.
8 20 42 18	6	11*t* 11	−1 0.14	−2 8.1	16 45 10.81	−3 14 19.7	9.012	0.617*n*	+1.89	−17.9	Nucleus. Brightness=10m.0

t signifies transits.

Mean Places of the Comparison-Stars for the Beginning of the Year.

*	α	δ	Authority		*	α	δ	Authority	
	h m s	° ′ ″				h m s	° ′ ″		
1	16 14 12.07	−8 51 39.3	A.G. Wien-Ottakring	5669	4	16 25 51.87	−7 19 40.1	A.G. Wien-Ottakring	5715
2	16 24 10.06	−7 56 12.7	A.G. Wien-Ottakring	5712	5	16 32 18.32	−5 24 31.2	A.G. Strassburg	5706
3	16 20 33.63	−7 46 29.2	A.G. Wien-Ottakring	5693	6	16 46 9.06	−3 11 53.7	A.G. Strassburg	5764

ELEMENTS OF COMET *b* 1914 (*ZLATINSKY*).

The following elements of Comet *b* 1914 (ZLATINSKY) have been computed by Professor CRAWFORD and Miss LEVY at the Students Observatory, Berkeley, California, from observations on May 16, 17, and 18.

T = May 8.38, 1914 G.M.T.
$\omega = 116° 22'$
$\Omega = 32\ 36$
$i = 112\ 59$
$q = 0.543$

CONTENTS.

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PRECESSION OF THE MARTIAN EQUINOXES,

By PERCIVAL LOWELL.

1. The solar annual precession for any planet is given very approximately, in consequence of the smallness of the planet's diameter as compared with its distance to the Sun, by the equation:

(1)

$$\text{Solar annual precession} = \frac{6\pi^2 . B . \cos I . n}{T\omega\,(1-e^2)^{\frac{3}{2}} . n + 1} \text{ (in radians)}$$

in which B = its momental function; $\frac{C-A}{C}$

I = the obliquity of its ecliptic, that is the angle between its equator and its orbital plane;

T = its sidereal year;

ω = its mean angular siderial velocity of rotation;

e = the eccentricity of its orbit;

n = the ratio of the Sun's mass to its own.

To reduce (1) to seconds of arc we multiply it by

$$\frac{180 . 60 . 60.}{\pi}$$

$$\text{giving (1)} = \frac{B . 9 . (60)^3 . \cos I . n}{T . (1-e^2)^{\frac{3}{2}} . n + 1} \text{ (in seconds of arc)}$$

when T is reckoned in sidereal days of the planet.

Except for *Jupiter* and *Saturn* $\frac{n}{n+1}$ is sensibly unity.

2. B, the momental function, is the moment of the impressed forces divided by the moment of inertia of the body. Its value depends upon the distribution of the matter composing the body. For two limiting cases the function may be calculated, to wit: for homogeneity on the one hand, and for complete central concentration on the other. The actual distribution must lie between the two.

For homogeneity the value of B is

$$B_h = \frac{a^2 - c^2}{a^2 + c^2}$$

where a and c are the long and short axes of the spheroid. If then we know the polar flattening we can find B_h. For the Earth the best value of the latter is $\frac{1}{297.5}$.

On the other hand if all the matter composing a planet were concentrated at its center, that infinitesimal kernel of exceedingly dense material would, as I have pointed out elsewhere*, be perfectly spherical. In consequence the momental function in that case or B_c would vanish. We have then two definite limits for B in the case of a planet corresponding the one to homogeneity, the other to central condensation. For the Earth their values are:

$$B_h = 0.0033749 \qquad \left(\text{polar flattening} = \frac{1}{297.5}\right)$$
$$B_c = 0.$$

and for *Mars*, taking $\frac{1}{200}$ for the amount of the oblateness $\eta = \frac{a-c}{a}$

$$B_h = .0050125$$
$$B_c = 0.$$

3. The next step is to find the actual B for the Earth. This we may obtain from the observed precession. For since in (1) I, e and T are known very approximately if we equate to it the observed solar precession we can deduce B. Now the solar precession is not directly furnished by observation. What observation gives is the general precession, from which must be subtracted the planetary precession to give the luni-solar one. This in its turn must be separated into its two constituents, a procedure possible by means of the theoretic values of each, since each involves B directly which therefore cancels out of the ratio.

The values of these several precessions as computed

* See *Phil. Mag.*, May, 1910, page 708.

by NEWCOMB, the result differing but little from L. STRUVE's for the luni-solar, are:

General precession $= 50''.2504$ a year
Planetary precession $= -\ .1127$ a year
Luni-Solar precession $= 50\ .3631$ a year

Now the ratio of the solar to the lunar annual precession is:

$$= \frac{T'^2\,(n'+1)\,(1-e'^2)^{\frac{3}{2}}}{T^2\,(1-\frac{3}{2}\sin^2 i)\,(1-e^2)^{\frac{3}{2}}}$$

in which n' is the ratio of the mass of the Earth to that of the Moon and i the inclination of the lunar orbit to the ecliptic. T' and T are the sidereal month and the sidereal year, respectively, expressed in terms of the sidereal day.

For the value of n' we have:

(1) from the constant of nutation $9''.21$ and the luni-solar precession $50''.36$ $n' = 81.58$ according to NEWCOMB.
(2) from the lunar inequality, taking the solar parallax at $8''.80$ $n' = 81.41$

The mean of the two gives $n' = 81.5$

Finally
$e = .0167468$ for 1910
$e' = .054908$ for 1910
$i = 5°\,8'\,48''$ for 1910

With these data we get:

solar annual precession $= 15''.993$
lunar annual precession $= 34\ .370$
luni-solar annual precession $= 50\ .363$

and equating the solar annual precession to (1)

$$B = .0032830$$

4. Now it is evident that *Mars* is nearer homogeneity than the Earth; because while the mean density of the Earth is 5.53 according to BOYS, that of *Mars* is but 3.93. The surface rocks of the Earth have a mean density of 2.65 and from what we know of the constitution of the planetary system including the mean density of meteors, we are justified in concluding the surface density of *Mars* not to differ greatly from our own. In consequence the gradient of increase of density inwards must be there less than here – or B_1 must more nearly approach B_{1h} than is the case with the Earth.

If the matter composing *Mars* were so arranged that the law of density were there what it is here, we should expect the same ratio to hold for B_1. Or B_1 would equal .0048761.

Consequently B_1 must lie somewhere between

.0050125
and .0048761

Furthermore from the fact that for the Earth where the surface density is one half the mean, the value of B approaches B_h so nearly, much more than half way from B_r to B_h, we perceive that it must do so increasingly for *Mars* where the ratio $\frac{\text{surface density}}{\text{mean density}} = .674$.

5. Our problem is therefore this: given three values of an unknown function to interpolate a fourth. Interpolation by differences is futile as the third differences are lacking and would be large.

Now we know that any quantity may be expressed as some function of another if the two be theoretically connected. Therefore

$$B = fx \quad \text{where } x = \text{ratio } \frac{\text{surface density}}{\text{mean density}}$$

Of this function we know that it is 0 when $x = 0$; that it is .0033749 when $x = 1$; and .0032830 when $x = \frac{1}{2}$. The simplest function that will satisfy these three values is:

$$B = C\,x^{\frac{1}{y}}$$

$$\text{or } \log B - \log C = \frac{\log x}{y}$$

whence y comes out 25.15.

Applying this to *Mars* we find from

$$C = .0050125$$
$$x = .674$$
$$B_1 = .004935$$

6. With this value for B_1 we are in position to calculate the solar annual precession on *Mars*, the other factors to the equation being known. From the latest determination at the Lowell Observatory

$$I_1 = 23°\ 30'$$

while

$$e_1 = .093318 \text{ for } 1910$$
$$T_1 = 669.60$$

Therefore the solar annual precession for *Mars:*

$$= \frac{B_1\,.\,9\,.\,(60)^3 \cos I_1}{T_1\,.\,(1-e_1^2)^{\frac{3}{2}}}$$

$= 13''.31$ in one Martian year
or $= 7\ .08$ in one terrestrial year

since the Martian sidereal year = 686.979 mean solar terrestrial days.

The great difference between this result and the one for the Earth is due to the greater distance of the Sun, the other factors being on the whole larger.

An interesting corollary follows from this investigation on the precession. Contrary to one's general ideas on the subject if the pole of the disturbed body coincide with the pole of the body's ecliptic this coincidence not only does not annul the precession but actually renders it a maximum.

This may be demonstrated as follows:
The amount of the motion of the pole of the disturbed's equator around the pole of its ecliptic, in a direction parallel to its ecliptic is:

$$\frac{6\,\pi^2 B\,.}{T\omega(1-e^2)^{\frac{1}{2}}}\;\frac{n}{n+1}\;D\,.\,\cos I\,.\;\sin I\;(C+l)\;-\;\text{terms}$$

which are periodic, dependent on the Sun's longitude and the longitude of the periplanetan l being the longitude of the disturber and C a constant. In the case of the Sun, I is the obliquity of the ecliptic.

To find the precession of the equinoxes we must divide this amount by sin I, giving for the precession

$$= a\,.\,\frac{\cos I\,.\sin I}{\sin I}\;(C+l)$$

Now let us examine the value at the limit when $I = 0°$. In this case the precession becomes

$$= a\,.\,\frac{0}{0}\,(C+l) \qquad \text{that is indeterminate.}$$

To evaluate it we differentiate both numerator and denominator according to the well-known method in cases of indetermination. This gives

$$= 2\pi a\,\frac{\cos^2 I - \sin^2 I}{\cos I}$$

in which making $I = 0°$ we find

$$= 2\pi a \cos I = 2\pi a$$

This is a maximum value for the function since cos I has here its greatest value, 1.

We thus see that in the case of coincidence of the planes of the orbit real or virtual of the disturber and of the equator of the disturbed, the value of the precession of the disturbed's equinoxes, far from being zero, has in fact its maximum amount. The physical explanation of this paradox is that though the amount of motion induced in one pole round the other parallel to its ecliptic is in truth nothing the pole of its ecliptic is also no distance from the planet's axial pole, giving for the precession the indeterminate form $\frac{0}{0}$, which analytics show has in this case in fact a finite value — and that value actually its maximum.

Lowell Observatory, May, 1914.

DISCORDANT MAGNITUDE DETERMINATIONS INDICATING POSSIBLE VARIABILITY,

By HEROY JENKINS.

In the course of a preliminary reduction of the Photometric work of the Department of Meridian Astronomy of the Carnegie Institution of Washington at San Luis a number of discordances were found which may be of interest to variable star observers. The list which follows contains stars for which the San Luis observations were discordant; stars whose magnitudes measured at San Luis differed considerably from those given by Harvard; and a few stars whose magnitudes differed from those given by GOULD. The San Luis observations given in the list have been reduced to the Harvard scale. The magnitudes taken from the Argentine *General Catalog*, the *Zone Catalog*, the Cordoba D M and the Cape Photographic D M have not been corrected. On one night So_{50} 1232 at $10^h\,14^m\,52^s$ was certainly fainter than 10.5 as it could not be seen, while the adjacent stars in the field were full brightness.

SUSPECTED VARIABLES.

Name	R.A. 1910	Decl. 1910	SAN LUIS	HARVARD	A.G.C.	Z.C.	C.D.	C.P.D.
	h m s	° ′						
L 58	0 17 41	−38 53.0	8.46 8.68 8.27	8.18 8.02 7.88	7¼	8.0	8.1	8.0
L 171	0 34 52	−74 27.2	7.83 7.97	7.50 7.51 7.55 7.50 7.41 7.26 7.22 7.32	7½	7.5	..	7.5
L 212	0 37 28	−83 31.7	8.56 8.55	9.26 9.26 9.06	9½	..	..	7.8
L 208	0 41 27	−54 49.6	9.62 9.55	9.00 9.00	8½	8.5	..	8.4
L 377	1 18 0	−52 32.9	8.60 8.71 9.07		7.2	8.5	..	8.1
L 461	1 24 44	−80 21.8	8.93 9.16		7	..	..	8.8
L 942	2 53 38	−36 47.7	8.10 7.82	7.36 7.52 7.41	7.4	7.7	7.4	7.3
CPD−24, 417	3 21 30	−24 47.9	9.96 9.63	9.10 9.10	..	8.8	9.0	8.6
L 1263	3 27 21	−82 34.7	8.54 8.81 8.71	8.07 7.97 8.07	7¼	..	.	7.8
Lal 8769	4 33 28	−23 13.8	8.42 8.15 7.77 8.05 8.06		8	8	7.8	6.8

Name	R.A. 1910	Decl. 1910	San Luis	Harvard	A.G.C.	Z.C.	C.D.	C.P.D.
	h m s	° ′						
L 1757	5 5 30	−59 59.7	7.24	7.92 8.02 6.82	7¼	8.0	..	8.4
L 1795	5 14 15	−39 23.1	7.73 7.11 7.28	6.72 6.63 6.79 7.16 7.18	7.2	8.0	7.5	6.9
L 1827	5 19 50	−31 49.7	7.40 7.99 8.12 7.60 8.50 7.44	7.30 7.78 7.27	7¾	7.6	7.5	7.6
L 2719	6 57 3	−80 1.1	8.03 8.39	8.99 8.79	8¼	8.5	..	8.9
Tay 3090	7 27 43	−34 47.6	7.29 7.45 7.28 7.07 6.93 7.08 7.04 7.62 7.56	8.15 7.65 7.51 7.64 8.01 7.35 7.46	7½	8.0	7.9	8.2
L 2883	7 31 56	−44 29.7	6.47 6.29		7½	7.5	7.1	8.4
L 2964	7 42 4	−39 37.8	6.78 7.47 7.95 7.41		7¾	8.0	8.0	8.5
Tay 3518	8 14 2	−59 46.7	7.34 7.24 7.58 7.44 7.40 7.47 7.76	7.99 8.39 7.89 7.59 8.09	7¾	7.5		8.9
L 3348	8 26 11	−30 1.4	7.60 7.52 7.27 7.52 6.74	8.24 7.74	8	8.0	7.7	8.8
So_{60} 1185	10 1 54	−55 19.7	7.65 7.76 7.99 8.08	8.34 8.84	8½	8.1	..	8.6
So_{60} 1232	10 14 52	−44 47.4	9.65 8.95		9½	9½	9.8	9.3
L 4647	11 8 26	−46 28.6	7.66 7.86	6.77 6.95	7	7	7.7	8.3
So_{60} 1086	12 49 8	−43 23.6	8.10 7.26 8.61		8¾	.	8.5	9.3
So_{60} 1494	16 47 8	−44 56.3	8.05 8.13 8.02 8.01 8.40 8.07 7.84 8.13	8.46 8.66	8¼	8.0	8.6	8.9
CD 11652	17 51 17	−50 8.1	9.48 8.96 8.86 9.14 9.03	9.29 9.99 10.19 10.09 9.79	..	9.5	9.5	9.6
Bb 6534	18 53 56	−65 2.6	7.89 7.35 7.48 7.68 7.23	8.05 8.15	8	8.0	..	7.7

Dudley Observatory, Albany, N.Y.

PHOTOGRAPHIC POSITIONS OF COMET 1911 *c* (*BROOKS*),

By WILLIAM O. BEAL.

The plates from which these positions have been obtained were exposed by Prof. F. P. Leavenworth and Mr. Alfred Davis at the observatory of the University of Minnesota with the 10½-inch equatorial. The scale of the plates is $1^{mm} = 1'$. The measurements were made with the Repsold measuring machine by the persons indicated in the column headed "computer."

The plates with long exposure times were originally made for the picture. On these the star images are trails from 1 to 3 mm. in length, and the positions obtained from them have a much smaller weight than those from the plates with short exposure.

The star places were taken from the A. G. Zones. The computers are designated as follows:

B = William O. Beal,
L = Greta Lagro,
T = Ella Thorp,
F = G. Lee Fleming,
Lv = F. P. Leavenworth.

	1911 Minneapolis M.T.	α 1911.0	δ 1911.0	log p Δ α	log p Δ δ	No. of Exposures	No. of Comp. Stars	Exposure Time	Computer
	d h m s	h m s	° ′ ″					m	
1	Sept. 15 10 1 31	16 44 58.30	56 34 47.8	9.858	0.278	1	4	30	B
2	Sept. 19 8 0 9	15 56 37.37	54 35 20.1	9.788	9.977	1	4	2	L
3	Sept. 21 7 43 5	15 34 12.68	53 9 13.6	9.787	0.137	1	4	2	B
4	Sept. 22 7 50 13	15 23 31.67	52 20 8.0	9.798	0.277	2	4	2	B
5	Sept. 22 7 50 18	15 23 31.83	52 20 9.1	9.798	0.277	2	4	2	F
6	Sept. 29 7 6 46	14 20 43.60	45 19 34.6	9.764	0.548	1	4	3	T
7	Oct. 7 6 59 39	13 29 17.66	35 20 10.5	9.703	0.745	4	6	2½	Lv
8	Oct. 8 7 5 56	13 24 4.41	33 58 17.3	9.690	0.768	3	6	2	Lv
9	Oct. 9 7 2 10	13 19 11.24	32 36 26.7	9.682	0.777	1	2	15	B
10	Oct. 10 6 57 11	13 14 31.11	31 13 9.6	9.678	0.772	1	3	26	B
11	Oct. 26 16 50 6	12 33 21.14	6 17 21.1	9.617*n*	0.780	1	4	19	B
12	Oct. 26 17 13 32	12 33 20.66	6 15 54.9	9.607*n*	0.775	1	4	6	B
13	Oct. 27 17 7 23	12 33 3.76	5 17 48.8	9.607*n*	0.780	1	4	25	B
14	Oct. 28 17 7 13	12 33 1.25	3 50 24.4	9.604*n*	0.783	1	2	41	B
15	Oct. 29 17 7 41	12 33 12.48	2 23 59.7	9.601*n*	0.786	1	2	34	B
16	Nov. 1 17 8 50	12 35 4.07	−1 46 20.1	9.594*n*	0.796	1	4	27	B
17	Nov. 4 17 28 21	12 38 32.22	−5 42 36.7	9.574*n*	0.808	2	2	5, 10	B

Minneapolis, Minn., June 2, 1914.

SUNSPOT OBSERVATIONS,

MADE AT BERWYN, PENN., WITH A 4½-INCH REFRACTOR,

By ALDEN W. QUIMBY.

1914	Time	New Grs.	Total Grs.	Total Spots	Fac. Grs.	Def.
Jan. 1	8	–	1	3	–	fair
2	8	–	1	5	–	“
3	8	–	1	1	–	poor
5	11	–	–	–	–	“
6	8	–	–	–	–	“
7	8	–	–	–	–	“
8	8	–	–	–	–	fair
9	8	–	–	–	–	“
10	8	–	–	–	–	“
11	12	–	–	–	–	“
12	8	–	–	–	–	“
13	8	–	–	–	–	“
14	8	–	–	–	–	“
15	8	–	–	–	–	“
16	8	–	–	–	–	“
17	2	–	–	–	–	“
18	8	–	–	–	–	“
19	8	–	–	–	–	poor
20	8	–	–	–	–	fair
21	8	–	–	–	–	“
22	8	–	–	–	–	“
23	8	–	–	–	–	“
25	8	–	–	–	–	“
26	8	–	–	–	–	“
27	8	–	–	–	–	“
28	8	–	–	–	–	“
29	8	–	–	–	–	“
30	8	–	–	–	–	“
31	8	–	–	–	–	“
Feb. 1	8	–	–	–	–	“
2	8	1	1	4	–	“
3	9	–	1	5	–	“
4	8	–	1	5	–	“
5	9	–	–	–	–	poor
7	2	–	–	–	–	fair
8	8	–	–	–	–	“
9	8	–	–	–	–	“
10	9	–	–	–	–	poor
11	8	–	–	–	–	fair
12	8	–	–	–	–	“
13	8	–	–	–	–	“
14	2	–	–	–	–	“
15	8	–	–	–	–	“
16	8	–	–	–	–	“
17	5	–	–	–	–	“
18	9	–	–	–	–	“
19	10	–	–	–	–	poor
20	11	–	–	–	–	“
21	8	–	–	–	–	fair
22	8	–	–	–	–	“
24	8	–	–	–	–	“
25	8	–	–	–	–	“
26	8	–	–	–	–	“
27	8	–	–	–	–	“
28	8	–	–	–	–	“
Mar. 2	10	–	–	–	–	poor
3	9	–	–	–	–	fair
4	8	–	–	–	–	“
Mar. 5	8	–	–	–	–	fair
7	9	–	–	–	–	“
8	8	–	–	–	–	“
9	9			–	–	“
10	8	–	–			“
11	3	–	–	–	–	v. poor
12	8	–	–	–	1	fair
13	8	–	–	–	1	“
14	8	–	–	–	–	“
15	8	–	–	–		“
16	7	1	1	2	1	“
17	7	–	1	2	1	“
18	11	–	–	–	–	poor
19	8	–	–	–	–	fair
20	10	–	–	–	–	“
21	8	–	–	–	–	“
22	11	–	–	–	–	“
23	8	–	–	–	–	“
24	8	–	–	–	–	“
25	8	–	–	–	–	“
26	8	–	–	–	–	“
27	3	–	–	–	–	“
28	8	–	–	–	–	“
29	2	–	–	–	–	v. poor
31	7	1	1	5	1	fair
Apr. 2	7	–	1	6	1	“
3	7	–	1	6	1	“
4	7	–	1	10	1	good
5	5	–	1	24	–	v. good
6	7	–	1	23	–	“
7	7	–	1	20	–	“
8	9	–	1	12	–	poor
9	8	–	1	6	–	fair
10	4	–	1	4	1	“
11	8	–	–	–	1	good
12	8	–	–	–	–	fair
13	8	–	–	–	–	“
14	8	–	–	–	–	“
16	2	–	–	–	–	poor
17	8	1	1	2	–	fair
18	5	–	1	10		v. good
19	7	–	1	6	–	good
20	3	–	1	1	2	fair
21	12	–	–	–	–	“
22	7	1	1	6	–	“
23	7	–	1	4	–	“
24	7	–	1	4	2	“
*25	12	–	–	–	–	v. poor
26	3	1	1	20	–	fair
27	4	1	2	32	1	good
28	6	–	2	24	1	fair
29	9	–	2	22	1	“
30	5	–	2	17	2	“
May 1	8		2	20	2	“
2	7		2	11	1	“
3	7	–	1	3	1	“
4	7	–	–	–	–	“
5	4	–	–	–	1	poor
May 6	4		–	–	1	fair
7	7	–	–	–	–	“
8	8	–	–	–	–	“
9	7	–	–	–	–	“
10	7		–			good
11	7	–	–	–	–	“
12	7	–	–	–	–	fair
13	5	–	–	–	–	“
14	7	–	–	–	–	good
15	7	–		–		“
16	7	–	–	–	–	“
17	7	–	–	–	–	“
18	7	–	–	–	–	“
19	7	–	–	–	–	“
20	7	–	–	–	1	“
21	7	1	1	3	1	“
22	5	–	1	5	–	fair
23	7	–	1	5	–	“
24	7	–	1	5	–	“
25	7	–	–	–	1	“
26	7	–	–	–	1	“
27	7	–	–	–	–	good
28	7	–	–	–	–	“
29	7	–	–	–	–	“
30	7	–	–	–	–	“
31	7	–	–	–	–	“
June 1	7	–	–	–	–	“
2	7	–	–	–	–	“
3	7	–	–	–	–	“
4	4	–	–	–	–	fair
5	7	–	–	–	1	good
6	7	–	–	–	1	“
7	5	–	–	–	1	“
8	7	1	1	4	1	“
9	7	–	1	4	1	“
10	11	–	1	4	–	fair
11	7	1	2	3	–	“
12	7	–	2	3	–	“
13	5	1	3	12	1	good
14	1	–	2	10	–	poor
15	9	–	2	28	1	good
16	7	–	1	20	–	fair
17	7	–	1	30	1	good
18	7	–	1	30	–	“
19	7	–	1	30	–	“
20	7	–	1	13	1	“
21	7	–	1	12	1	poor
22	2	–	–	–	–	fair
23	7	–	–	–	–	“
24	7	–	–	–	–	“
25	7	–	–	–	–	“
26	7	–	–	–	–	“
27	7	–	–	–	–	“
28	4	–	–	–	1	poor
29	7	–	–	–	–	fair
30	7	–	–	–	–	“

FURTHER DATA BEARING UPON THE REALITY OF THE ANTAPEX GROUP, *A. J.*, 635–636,

BY BENJAMIN BOSS.

In *A. J.* 635–636 a group of stars has been pointed out with a convergent at R. A. = 107°, Decl. = −40°. Because of the proximity of this convergent to the antapex of solar motion the group was called the Antapex Group. On page eighty-nine of the number referred to a comparison was made between the observed mean proper-motion of groupings arranged according to distance from the convergent, and the value of the proper-motion computed on the basis of true parallelism of motion of the group. The agreement was perfect.

It may now be interesting to treat the few stars of the group whose radial velocities are given. If once more it is assumed that perfect parallelism of motion exists in this case, we have the means of determining the group velocity from the observed radial velocities, and reversing the process we are able to obtain computed individual velocities from the group velocity.

Radial velocities were given for seven stars. Six of them are contained in the table, where the first column gives the number of the star in the *Preliminary General Catalogue*, the second column the observed radial velocity, the third the computed radial velocity, and the fourth the residual computed minus observed.

P. G. C.	ρ Obs.	ρ Comp.	C − O
6031	− 3.7	− 7.4	−3.7
101	+12.	+10.7	−1.3
107	+ 9.	+10.8	+1.8
422	− 1.2	− 1.6	−0.4
885	+17.	+16.3	−0.7
2974	+ 6.8	+ 4.0	−2.8

The computed values were based upon a group velocity of +21 kilometers relative to the Sun. One star *P. G. C.* 100 (β^1 *Tucan.*), was omitted. Its radial velocity 0 was based upon only one observation. The agreement between computed and observed radial velocities is very fair, and lends considerable more reality to the group.

The suspicious agreement between the vertex of the group and the antapex of solar motion taken in conjunction with the group velocity indicates that we are dealing with stars practically at rest in space.

As far as I am aware only two of the stars have measured parallaxes. *P. G. C.* 5944 is given an observed parallax of +″.026 by FLINT, while the value computed on the hypothesis of group motion is +″.024. The parallax of *P. G. C.* 2987 was found to be +″.041 by JOST, while the computed value amounts to +″.022. It is unfortunate that the computed parallaxes of these stars are so small, since it deprives us of an additional means of testing the reality of the group.

ERRATA IN BOSS'S PRELIMINARY GENERAL CATALOGUE,

BY ARTHUR J. ROY.

Through the kindness of some of our colleagues in furnishing us with a list of errata in the *Preliminary General Catalogue*, and through the discovery of additional errata by members of the staff, we have become possessed of a list of errors and typographical faults whose correction may save the reader considerable trouble. The list follows:

Page VII line Camb. 30, for Vol. X read Vol. XI.
VIII line Saff. 85, for 1875 read 1885.
XIX fourth line from bottom of page for $0^{m}.36$ read 0.36.

Page XXXII line 27 for *Pi* 1900 read *Pi* 1800.

Star No. 816 for $+^{s}.0028$ read $+^{s}.0034$.
1641–2 both notes pertain to 1642.
2033 for ″.008 read −″.008.
2087 for *a Puppis* read L 3044.
3064 for −5° read −75°.
3356 for $22^{s}.16$ read $23^{s}.16$.
3693 for L 5920 read L 5921.
3885 for −″.013 read +″.013.
4493 for *Sagitarii* read *Sagittarii*
4936 for $+^{s}.0025$ read $+^{s}.0027$.
4955 for −6″.853 read +6″.853.
5349 for $-^{s}.0002$ read $+^{s}.0001$.
5660 for 343° read 243°, foot note.

Page 266 line 11 from bottom of page for DOBERK read DOBERCK.

Page 267 line 4 for a^2 read a^1.
267 line 6 for a^1 read a^2.
274 line 28 for −1.33 read −2.33.
279 line 27 insert comma (,) after "right-ascension."
280 in formula for e_2 read $(e_1)^2$.
280 tenth line from bottom for ±″30 read ±″.30.
340 line 1 for "1880 to 1884" read 1880 and 1884.

THE TAURUS CLUSTER,

By JOHNSON O'CONNOR.

For over a year I have worked during my spare moments on the motions of groups of stars. It may be interesting, after the recent announcement from Dr. SLIPHER, that the nebula in *Virgo* is rotating, to summarize briefly what I have so far found. In 1908 (*Ast. J.* 604) LEWIS BOSS showed that thirty-eight stars, within an area in the sky of about thirty degrees diameter, have motions in space practically equal and parallel to one another. This means that the projections of these motions — or the proper motions which we measure — must all converge at a single point. This point Professor BOSS gives for 1875 as R.A. $6^h\ 7^m.2$; Dec. $+6°\ 56'$. I separated the motion of each star into two components, one on the line from the star to the convergent, the other perpendicular to this. If it is true that the motions in space are parallel to one another, this last component, across the line to the convergent, should be zero, except for errors of observation, regardless of the parallaxes of the stars. The other component, that toward the convergent, depends upon the parallaxes; and as these are unknown, or not known accurately enough, the length cannot be computed. Column I of the table gives the *Preliminary General Catalogue* number of each star. Column II gives the component of the star's motion perpendicular to the line from the star to the convergent. Columns III and IV give the probable errors as computed by Professor Boss and given in the *P. G. C.* Although about half of the residuals of Column II are larger than the probable errors given, and several are twice as large, there is little in this fact to show that they are not errors of observation. If however, the stars are arranged as in the table, according to their increasing position angle about the point R. A. $4^h\ 20^m$, Dec. $14°.8$, near the center of the cluster the positive and negative residuals, instead of being distributed about evenly, fall into an arrangement too definite to be due to errors of observation.

I	II	III Dec.	IV R. A.	V Position Angle
	″	″	″	°
1040	.60	.47	.52	2
1044	.63	.17	.15	4
1045	.11	.23	.26	21
1046	.35	.24	.28	28
1054	.14	.39	.45	36
1055	−.04	.29	.33	52
1143	.67	.30	.33	57
1058	−.08	.38	.36	58
1086	−.03	.51	.44	70
1090	.60	.41	.34	74
1184	.82	.47	.44	83
1067	.00	.28	.26	97
1226	.50	.36	.38	116
1114	.02	.54	.60	132
1087	.98	.35	.44	133
1043	.48	.35	.38	136
1056	.36	.37	.39	147
1092	−.07	.46	.58	157
1047	.15	.34	.38	168
961	.00	.40	.51	201
1013	−.82	.34	.33	231
1004	−.51	.33	.36	240
1007	−.76	.40	.40	270
980	.17	.36	.38	275
892	−.28	.40	.48	284
1000	.08	.16	.18	287
919	−.25	.44	.57	295
952	−.85	.31	.30	315
1018	−.02	.42	.42	331
1031	.10	.46	.45	335
1017	.08	.18	.18	336
1034	.27	.40	.39	340
991	.45	.29	.30	341
1022	−.37	.29	.32	342
1029	1.00	.28	.30	352
1026	.02	.23	.30	356
1027	−.46	.38	.48	356
1033	.46	.27	.38	358

Position Angle	Number of + Residuals	Mean of + Resid.	Number of − Residuals	Mean of − Resid.
333°—153°	20	″.43	5	″.13
153°—333°	3	″.20	8	″.44

One's first thought is that this is due to a wrong convergent point. Professor Boss found the convergent by least squares. Possibly it would be better to make the sum of the squares of the residuals larger, if by this one could do away with the arrangement of plus and minus. Eight convergent points were therefore chosen, surrounding the first, and increments computed to be added to the old motions to give the new. If the direction of the new convergent from the old were right, regardless of its distance, within certain limits, the increments should have the same signs as the residuals. Neither from these, nor by interpolating between them, could I find a new convergent which would justify the increase in the Σvv.

If, as it seems, the residuals are due partly to actual motions, opposite in direction on opposite sides of the center, the stars are either separating into two groups or are revolving about a center near R. A. $4^h\ 20^m$, Dec. $14°.8$. If the last is true the stars near position angles 153° and 333°, where the residuals change sign, should have residuals numerically less than stars 90° from these points. If, however, the stars are separating into two groups, the residuals will be approximately the same size everywhere. The mean of the three residuals between positions angles 138° and 168° is ″.19, the mean of the six between 318° and 348° is ″.22. Ninety degrees from these points there

are five residuals between 48° and 78°, two between 228° and 258°; and their means are respectively ″.28 and ″.66. This gives a mean of ″.21 for the residuals near 153° and 333°, and of ″.39 for those ninety degrees from these points. Everything, so far at least, points to the fact that the stars are revolving about a common center.

If, however, the stars are moving in circles whose plane is perpendicular to the line of sight the signs of the residuals should change where the angles between the direction of the residual motions and the directions of the tangents to the circle are 90° and 270°. Instead the signs change where these are 60° and 240°. In order that this shall be so the motions of the stars in the plane perpendicular to the line of sight must be in an ellipse whose eccentricity is more than .8 and whose major axis lies between position angles 290° and 340°. This means that if the stars are moving in circles, or nearly so, they are inclined about 50° or more to the tangent plane.

I have tried to go further and have made several assumptions, that the stars were revolving under the law of gravity, that they were revolving as a solid mass, and that they were each moving the same amount. But not one of the three fitted, even making allowances for errors of observation. There was but one arrangement which seemed to fit at all. That they follow this I cannot prove and I wish to state it only as a possibility. There is a group, spherical in shape, clustered closely around the center and revolving as a solid mass. The rest are scattered but all lie in a plane passing through the center group, and move, not as a solid mass, but the outside ones less than the inside

To summarize briefly, although the residuals published by Professor Boss are small compared with the probable errors, their signs follow an arrangement which suggests that the thirty-eight stars besides moving in parallel lines are also revolving about a common center.

NEW LONG PERIOD VARIABLE IN *CARINA*,

By ALEX W. ROBERTS.

While making observations of *Z Carinæ* the variability of a star, whose approximate position for 1875 is

$10^h\ 9^m\ 35^s$
$-58°\ 24'$

has been detected.

The variation has been confirmed at Harvard through Professor Pickering kindly examining the photographic estimates of the star.

The following are the Lovedale measures of the star:

				M					M
1908					1913				
Jan.	5	10	10	9.30	Mar.	1	7	0	10.73
	8	8	12	9.40		10	6	45	10 70
1912						15	6	15	10.90
Dec.	15	7	40	9.50	April	2	10	30	11.00
	16	11	25	9.60		5	7	0	11.00
1913						6	5	50	11.00
Jan.	31	10	5	9.70		11	8	50	11.20
Feb.	10	7	50	10.30		29	7	0	<11.30
	14	7	0	10.45	May	5	7	0	<11.30

				M					M
1913					1914				
May	12	5	35	<11.30	May	16	8	55	9.75
June	3	5	28	<11.30		19	7	50	9.75
	21	5	20	<11.30		21	6	20	9.70
	27	6	33	<11.30		22	5	20	9.70
Aug.	4	7	20	<11.30		23	6	30	9.90
						24	6	0	9.75
1914						30	6	15	10.25
Feb.	15	6	10	10.10	June	12	6	0	10.60
Mar.	27	7	0	9.10		16	7	20	10.80
Apr.	20	6	30	9.25		20	7	14	10.90
	28	10	25	9.50		21	5	0	11.00
May	2	5	5	9.80		22	6	30	11.10
	6	6	15	9.85		23	4	50	11.10

The magnitudes are only roughly reduced and may be $0^M.2$ in error.

A period of 450 days will prove to be not far from the true duration of variation.

Lovedale, 24 June 1914.

EDITORIAL.

We take great pleasure in announcing the appointment of Prof. E. E. Barnard, Dr. F. R. Moulton, and Dr. R. S. Woodward as associate editors of the *Astronomical Journal*. Prof. Ernest W. Brown continues to serve as associate editor.

CONTENTS.

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DEVIATIONS OF FALLING BODIES,

By WM. H. ROEVER.

[This article consists of two parts. Part I contains a discussion of Dr. R. S. Woodward's recent work. Part II contains a new treatment of the problem. The author wishes it to be understood at the outset that the frequent use of Dr. Woodward's name in references to his work is made purely in the interest of accuracy and precision. The same consideration accounts for the frequent use of the personal pronoun.]

PART I.

Introduction.

In an article recently published in the *Astronomical Journal* (Nos. 651–652) under the title: "The Orbits of Freely Falling Bodies,"* Dr. R. S. Woodward treated a problem which had already been solved by me, by two different methods, in Vols. XII and XIII of the *Transactions of the American Mathematical Society*.†

In this article claim is made (page 18, middle of first column) "to carrying the order of approximation in the solution of the problem one step beyond that hitherto attained."

In the following pages it will be shown that this claim has not been substantiated — Dr. Woodward not having carried the order of approximation in the solution of the problem as far as was done in my papers, published before his — and furthermore that the results which Dr. Woodward gives for the meridional deviation are not the ones which correspond to his assumptions.

In his summary on p. 29 Dr. Woodward asserts that, "The problem of falling bodies has hitherto been inadequately treated by reason of neglect of the effect of the square of the angular velocity of the Earth," and at the top of the first column of p. 22 he says that "The equations of Gauss, Poisson, and all subsequent writers, so far as I am aware, are equivalent in their first members to the first members of (25), if terms in ω^2 are neglected." Concerning these assertions, I should like to point out that a superficial reading of the equations of motion which are expressed in terms of the potential function of the field of force which determines weight, might lead one erroneously to the conclusion that ω^2 was neglected, when in reality it was not. For instance, Dr. Woodward's differential equations (25_1) :

$$\frac{d^2\rho}{dt^2} - 2\omega\frac{d\eta}{dt} - \rho\omega^2 = \frac{\partial V}{\partial\rho}, \quad \frac{d^2\eta}{dt^2} + 2\omega\frac{d\rho}{dt} - \eta\omega^2 = \frac{\partial V}{\partial\eta}, \quad \frac{d^2\sigma}{dt^2} = \frac{\partial V}{\partial\sigma}, \tag{α}$$

are expressed in terms of the potential function V of the field of force due to gravitational attraction alone. If we denote by W the potential function of the field of force which determines weight, then W and V are connected by the relation

$$W = V + \frac{\omega^2}{2}\left(\rho^2 + \eta^2\right), \tag{β}$$

and hence

$$\frac{\partial W}{\partial\rho} = \frac{\partial V}{\partial\rho} + \omega^2\rho, \quad \frac{\partial W}{\partial\eta} = \frac{\partial V}{\partial\eta} + \omega^2\eta, \quad \frac{\partial W}{\partial\sigma} = \frac{\partial V}{\partial\sigma}. \tag{γ}$$

If we now substitute for the derivatives of V their expressions in terms of W, the above equations become

$$\frac{d^2\rho}{dt^2} - 2\omega\frac{d\eta}{dt} = \frac{\partial W}{\partial\rho}, \quad \frac{d^2\eta}{dt^2} + 2\omega\frac{d\rho}{dt} = \frac{\partial W}{\partial\eta}, \quad \frac{d^2\sigma}{dt^2} = \frac{\partial W}{\partial\sigma}, \tag{δ}$$

These equations do not explicitly contain ω^2 but nevertheless, they are the equivalents of the equations first written and not the approximations thereto made by neglecting ω^2. In my second paper I wrote my differential

* An article of the same title was published by Dr. Woodward in *Science*, Vol. 38 (New Series), pp. 315–319.

† The first of these articles is entitled "The Southerly Deviation of Falling Bodies," and occupies pp. 335-353 of Vol. XII, the second is entitled "The Southerly and Easterly Deviations of Falling Bodies for an Unsymmetrical Gravitational Field of Force," and occupies pp. 469-490, Vol. XIII. I shall call these my first and second papers, respectively.

equations of motion in the latter form, *i.e.* in terms of the potential function W. Hence my treatment was not inadequate by reason of the neglect of ω^2. In fact, my differential equations (2) on p. 177 of my second paper are the same as equations (δ) above. Dr. WOODWARD's form of these equations is given by me several lines above my equations (2), my letters x, y, z, U, corresponding to his letters ρ, η, σ, V.

Similarly, if Dr. WOODWARD's equations (25):

$$
(\epsilon)\quad
\begin{aligned}
&\frac{d^2\xi}{dt^2} - 2\omega \sin\varphi \cdot \frac{d\eta}{dt} - \omega^2\rho \sin\varphi = \frac{\partial V}{\partial \xi}\,,\\
&\frac{d^2\eta}{dt^2} + 2\omega\left(\sin\varphi \cdot \frac{d\xi}{dt} + \cos\varphi \cdot \frac{d\zeta}{dt}\right) - \omega^2\eta = \frac{\partial V}{\partial \eta}\,,\\
&\frac{d^2\zeta}{dt^2} - 2\omega \cos\varphi \cdot \frac{d\eta}{dt} - \omega^2\rho \cos\varphi = \frac{\partial V}{\partial \zeta}\,.
\end{aligned}
$$

be expressed in terms of the potential function W, they assume the form:

$$
(\iota)\quad
\begin{aligned}
&\frac{d^2\xi}{dt^2} - 2\omega \sin\varphi \cdot \frac{d\eta}{dt} = \frac{\partial W}{\partial \xi}\,,\\
&\frac{d^2\eta}{dt^2} + 2\omega\left(\sin\varphi \cdot \frac{d\xi}{dt} + \cos\varphi \cdot \frac{d\zeta}{dt}\right) = \frac{\partial W}{\partial \eta}\,,\\
&\frac{d^2\zeta}{dt^2} - 2\omega \cos\varphi \cdot \frac{d\eta}{dt} = \frac{\partial W}{\partial \zeta}\,.
\end{aligned}
$$

The latter equations are equations (5) on p. 481 of my second paper. They are not only not approximations to Dr. WOODWARD's equations obtained by neglecting ω^2, but they are more general than Dr. WOODWARD's equations for the reason that they hold for a general asymmetric distribution of the Earth's gravitating matter, whereas Dr. WOODWARD's equations hold for the less general case of a distribution of revolution, *i.e.* a distribution for which $\partial V/\partial\lambda = 0$, where λ denotes longitude. (See footnote to equations (8) of this paper.)

Besides the differential equations of motion already written, Dr. WOODWARD gives the equations of motion in terms of the polar coördinates r, ψ, λ (his equations (24)), and I give on p. 347 of my first paper the equations of motion in terms of cylindrical coördinates. These equations are equations (15) and (13) respectively of this paper. The latter equations, *i.e.* (13), are simpler, and probably more easily integrated, than the former. The various coördinates used by Dr. WOODWARD are shown in Fig. 3 of this paper, and those which I use are shown in Fig. 2*. Dr. WOODWARD practically assumes a distribution of revolution by his statement, below his equations (3), that "λ_1 and μ_1 refer to P_0 as well as to P_1 since both are in the same meridian plane." I have not made this restriction.

Dr. WOODWARD does not explicitly define his terms meridional and easterly deviations. He states precisely below his equations (2) how the "point P_1 is located with reference to the initial point P_0." Below his equations (18) he implies that these deviations are the values which his variables ξ and η have, respectively, for the particular value of the time t, for which $\zeta = 0$. These are the definitions which I give in § 1 on p. 473 of my second paper, my letters P_0 and R corresponding respectively to his letters P_0 and P_1. Definitions 1 and 2 in Part II, § 1, of this paper are very precise definitions of these terms. See also Fig. 1.

Among "The defects in previous investigations which need to be remedied," Dr. WOODWARD mentions in the first column p. 18 "(*a*) inadequate expressions for the gravitational potential of the Earth for points outside its surface; (*b*) unjustifiable neglect of terms in the complete differential equations of motion of the falling body; (*c*) neglect of the distinction between geocentric and geographic (or astronomical) latitude." By defect (*b*) Dr. WOODWARD probably means the neglect of the square of the angular velocity, that is, of ω^2, and of this neglect I have already shown myself to be not guilty. A careful reading of my papers will show that they do not contain defect (*c*). Concerning defect (*a*) I should like to say that in Part II, § 4, of this paper, I show that the potential function which I use under assumption 4, of my first paper, (p. 342), and corresponding to which I derive the formula for the southerly deviation which is given at the bottom of p. 344 of that paper, is the same as Dr. WOODWARD's potential function (46). Therefore I am also not guilty of defect (*a*) if Dr. WOODWARD's potential function (46) is to be regarded as an adequate expression for the gravitational potential function of the Earth. Concerning the adequacy of this potential function I will say something in the sequel. (See the REMARK which follows.)

I have now shown that the mathematical problem treated by Dr. WOODWARD is the same as one previously treated by me, that is, that (1) the definitions of the deviations, (2) the differential equations of motion, (3) the potential function are the same in both treatments; and furthermore, that my treatment does not contain the defects which Dr. WOODWARD attributes to investigations made before his. Notwithstanding these facts, Dr. WOODWARD's results differ from mine. Statements of the results obtained in the two treatments are given below.

*In Fig. 3 the path c of the falling particle is shown by a curve which, in the neighborhood of P_0, resembles an ellipse. This is the case because Fig. 3 represents a view taken from a point fixed with respect to the axes $0-x$, y, z, which axes do not rotate with respect to the inertial axes of the solar system. In Fig. 2 the path Γ of the falling particle is shown by a curve which has a cusp at P_1, and this is the case because Fig. 2 represents a view taken from a point fixed with respect to the solid part of the rotating Earth.

Dr. Woodward confines his treatment to a distribution of revolution and furthermore, to the particular distribution of revolution which corresponds to his potential function (46). For the easterly deviation he derives a formula which is practically the same as that given by Gauss for this deviation. For the meridional or southerly deviation he does not give an explicit formula, stating however, that it is always away from the equator (statement 3 in summary on p. 29) and that it varies as the square of the time of fall (Science article, middle second column, p. 318). For the particular data: height of fall $h = 49024$ cm. and astronomical latitude $\phi = 45°$, he finds as the numerical value of the meridional deviation 3.03 cm., *i.e.*, $\xi = -3.03$ (see top of second column, p. 25, of his *A.J.* paper). Incidentally he remarks (end of second paragraph, first column, p. 19) that the point P_1 is nearer to the equator than the point T, in which the normal at P_0 pierces the plane tangent to the equipotential surface through P_1.

In my first paper I confine my treatment to a distribution of revolution and to the southerly deviation, for which I derive the general formula :

$$S.\ D. = \left[2\omega^2 \sin 2\phi - 5\frac{\partial g}{\partial \xi}\right]\frac{h^2}{6g}\,, \tag{κ}$$

in which $S.\ D.$ denotes southerly deviation of the falling particle, ω denotes angular velocity of Earth's rotation, h denotes the height of fall, and ϕ, g, $-\partial g/\partial\xi$ denote values, at the place of observation, of the astronomical latitude, the acceleration due to weight, and the derivative of the acceleration with respect to the latitude, measured in linear units along a level surface. (See formula (3), p. 338, where η corresponds to $-\xi$ here used). Then I find the special forms which this general formula assumes for four different potential functions. Thus I obtain for Gauss' assumption (that the lines of force of the weight field have no curvature, *i.e.*, that $\partial g/\partial\xi = 0$), Gauss' formula $\{S.D. = \frac{1}{3}\omega^2 \sin 2\phi\ .\ h^2/g\}$ for the southerly deviation; for two assumptions made by M. le Comte de Sparre I obtain results obtained by him*, and for the potential function which Dr. Woodward uses (his formula (46)) I derive the special formula:

*Comte de Sparre uses a different definition for the southerly deviation than the one here used. He measures this deviation from the vertical of the initial point P and not from the vertical of P_1 as Dr. Woodward and I do. Various definitions have been given for the southerly deviation, but the one which I have adopted is the one directly measured in experiment. M. Fouché, in Vol. XXXIII, of the *Bulletin de la Société Mathématique de France*, gives four different definitions for southerly deviation, but he does not show how to compute any of these deviations. In an unpublished note which I communicated to Professor F. R. Moulton in a letter dated Feb. 21, 1911, I derived formulas for each of the four types of deviation given by M. Fouché.

$$S.D. = \frac{1}{2}\left(4\sigma_1 - \frac{5}{3}\epsilon\right)\sin 2\psi_0\ .\ \frac{h^2}{r_0}\,, \tag{λ}$$

where ψ_0 and r_0 denote the values at the initial point P_0, of the geocentric latitude and distance from Earth's center, and σ_1 and ϵ are constants which have the values

$$\sigma_1 = .00346\ , \qquad \epsilon = .00167\ . \tag{μ}$$

(See formula at bottom of p. 344 where ϕ_0 and ρ_0 correspond to ψ_0 and r_0 here used).

In my first paper I also point out the necessity of using an approximation to the potential function which, at the initial point, has not only the same first derivatives, but also the same second derivatives as the function approximated. Also the advantage of using the definition of southerly deviation which I adopted, in order to be able to compare theory with observation. (See introduction p. 335).

In my second paper* I treat the problem under the general assumption of an asymmetric distribution and find for the deviations the general formulas :

$$\begin{aligned} E.D. = {} & \frac{2}{3}\sqrt{2}\ .\ \omega\cos\phi\ .\ \frac{h^{3/2}}{g^{1/2}} - \frac{5}{6}\left(\frac{\partial g}{\partial\eta}\right)_0\frac{h^2}{g} \\ & + \frac{\sqrt{2}}{30}\ .\ \omega\left[3\cos\phi\ .\left(\frac{\partial g}{\partial\zeta}\right)_0 - 2\sin\phi\ .\left(\frac{\partial g}{\partial\xi}\right)_0\right. \\ & + 2\cos\phi\ .\left(\frac{\partial^2 W}{\partial\eta^2}\right)_0 + 4\,\omega^2\,(5\cos^3\phi \\ & \left. - 2\cos^2\phi)\right]\frac{h^{5/2}}{g^{3/2}}\,, \end{aligned} \tag{ν}$$

$$\begin{aligned} S.D. = {} & \left[2\,\omega^2 \sin 2\,\phi - 5\left(\frac{\partial g}{\partial\xi}\right)_0\right]\frac{h^2}{6g} \\ & + \frac{\sqrt{2}}{15}\,\omega\left[\cos\phi\ .\left(\frac{\partial^2 W}{\partial\xi\partial\eta}\right)_0 - 9\sin\phi\ .\left(\frac{\partial g}{\partial\eta}\right)_0\right]\frac{h^{5/2}}{g^{3/2}}\,, \end{aligned}$$

in which $E.D.$ and $S.D.$ denote easterly and southerly deviations respectively, ω, h, ϕ, g have the same meanings as above. W denotes the potential function of the field of force which determines weight, and ξ, η, ζ represent distances measured along the north-and-south line, the east-and-west line, and the vertical, respectively, at the initial point, these distances being positive to south, east and zenith, respectively, and hence

$$g = -\left(\frac{\partial W}{\partial\zeta}\right)_0\,, \qquad \left(\frac{\partial g}{\partial\xi}\right)_0 = -\left(\frac{\partial^2 W}{\partial\zeta\partial\xi}\right)_0\,,$$

$$\left(\frac{\partial g}{\partial\eta}\right)_0 = -\left(\frac{\partial^2 W}{\partial\zeta\partial\eta}\right)_0\,, \qquad \left(\frac{\partial g}{\partial\zeta}\right)_0 = -\left(\frac{\partial^2 W}{\partial\zeta^2}\right)_0\,.$$

where the subscript $_0$ denotes that the particular value has been taken which corresponds to the initial point P_0. For the special case of a distribution of revolution

$$(\partial^2 W/\partial\xi\partial\eta)_0 = 0 \quad \text{and} \quad (\partial^2 W/\partial\zeta\partial\eta)_0 = 0\,.$$

* *T.A.M.S.*, Vol. XIII, pp. 469–490.

and the second terms in the expressions for *E.D.* and *S.D.* drop out.

In my second paper I also call attention to the fact that the derivatives

$$\frac{\partial g}{\partial \xi} = -\frac{\partial^2 W}{\partial \zeta \partial \xi}, \quad \frac{\partial g}{\partial \eta} = -\frac{\partial^2 W}{\partial \zeta \partial \eta}, \quad \frac{\partial^2 W}{\partial \xi \partial \eta},$$

which enter into the early terms of the expressions for *E.D.* and *S.D.* can be determined experimentally by a method due to the Hungarian physicist, BARON ROLAND EÖTÖS. In Table II of that paper (which is also given in this paper, Pt. II, §4), where x, y, z, correspond to $-\xi, \eta$ and $-\zeta$ here used, experimentally determined values of these derivatives are given. The Table shows that these values exceed many times those

$$\left(-\frac{\partial g}{\partial \xi} = 8.14 \times 10^{-9} \sin 2\phi\ , \quad \frac{\partial g}{\partial \eta} \equiv 0\ , \quad \frac{\partial^2 W}{\partial \xi \partial \eta} \equiv 0\right)$$

which correspond to Dr. WOODWARD's potential function (46). In closing the introduction to that paper I state that "*The surprising conclusion is reached that known local irregularities in the Earth's gravitational field of force* (*caused by the presence of mountains or of mineral deposits or even of large buildings or tunnels*) (1) *influence the southerly deviation to the extent of from* -16 *to* $+40$ *times the amount which is given by the formula of* GAUSS *for the southerly deviation*" ($S.D. = \frac{1}{3}\omega^2 \sin 2\phi \cdot h^2/g$). "(2) *affect the easterly deviation by amounts which are comparable with the effect which* GAUSS *finds is due to air resistance.*"

REMARK. In view of this conclusion and on account of the fact (which I have already shown in my first paper) that corresponding to Dr. WOODWARD's potential function (46) the southerly deviation is about $4\frac{1}{2}$ times as great as that given by GAUSS' formula, it follows that, for the southerly deviation, Dr. WOODWARD's potential function (46) is quite inadequate. In fact the conclusion is sufficient to prove the inadequacy of any formula for the meridional (southerly) deviation of freely falling bodies which is based on a particular potential function, and to show *that, in the future, experiments for the determination of the deviations of freely falling bodies should be preceded by preliminary experiments for the determination of the local values of the derivatives* $\frac{\partial g}{\partial \xi}$, $\frac{\partial g}{\partial \eta}$, $\frac{\partial^2 W}{\partial \xi \partial \eta}$. These values of these derivatives when used in formulas (2) above take into account the effect of local irregularities in the Earth's field of force. However, formulas (ν) do not take into account the effects of air resistance, air currents, the actions of the Moon and Sun and the weight of the string which supports the plumb-bob. (See Pt. II, §4.)

In §1, Part I, of this paper, I show that if Dr. WOODWARD's first method of solution *i.e.* the one given under the heading: "*Integration of Equations* (24_1)" be carried through without making approximations, there will result for the easterly and southerly deviations the first terms of formulas (ν). More specifically, I show that if the solution r, ψ, λ of Dr. WOODWARD's equations (24) be obtained in powers of the time t, inclusive of the fourth power, and if then the transformation from the variables r, ψ, λ to the variables ξ, η, ζ be made by his exact relations (8) instead of by his approximate relations (18) and (19), and finally if it be recognized that the coefficient of t^2 in the development of ξ contains as a factor the quantity h (height of fall), which itself is proportional to the square of the time of fall, then there results for the easterly deviation the formula of GAUSS for that deviation and for the southerly deviation my formula (κ).

In §2, Part I, of this paper, I show (1) that in the integration of his differential equations (25_1) Dr. WOODWARD does not use the potential function (46) which he sets out to use, but that, on account of the assumption that the variable quantities ν_1^2 and ν_2^2 in his equations (51) are constants, he approximates his potential function (46), for which,

$$-\frac{\partial g}{\partial \xi} = \frac{2\sigma_1 - \epsilon}{r}\, g_a \sin 2\psi = 8.14 \times 10^{-9} \sin 2\phi\ , \qquad (o_1)$$

by a function for which

$$-\frac{\partial g}{\partial \xi} = \frac{\nu_2^2 - \nu_1^2}{2} \sin 2\phi = 5.13 \times 10^{-9} \sin 2\phi\ ; \qquad (o_2)$$

(2) that if his exact solutions ρ, σ, (56) (57) of his equations (51), in which ν_1^2 and ν_2^2 are regarded as constants, be transformed to the coördinates ξ, ζ by means of his exact equations (10), then there results for the southerly deviation, not the formula:*

$$S.D. = \left[2\,\omega^2 + 5\,\frac{2\sigma_1 - \epsilon}{r}\, g_a\right] \sin 2\phi\ .\ \frac{h^2}{6g} \qquad (\pi_1)$$
$$= 51.33 \times 10^{-9} \sin 2\phi\ .\ \frac{h^2}{6g}\,,$$

which corresponds to his potential function (46), but the formula:

$$S.D. = \left[2\omega^2 + 5\,\frac{\nu_2^2 - \nu_1^2}{2}\right] \sin 2\phi\ .\ \frac{h^2}{6g}$$
$$= 36.29 \times 10^{-9} \sin 2\phi\ .\ \frac{h^2}{6g}\,, \qquad (\pi_2)$$

*This is practically the same as formula (λ). For the coefficient of $h^2 \sin 2\phi$ may be written

$$\frac{1}{6}\,\frac{1}{r}\left[2\,\frac{\omega^2 r}{g} + 5\,(2\sigma_1 - \epsilon)\,\frac{g_a}{g}\right]$$

where g_a/g is nearly unity and $\omega^2 r/g$ is nearly equal to σ_1. Hence this coefficient takes the form $\frac{1}{3}\,\frac{1}{2r}\left[12\sigma_1 - 5\epsilon\right]$ and thus we get the formula

$$\frac{1}{2}\left(4\sigma_1 - \frac{5}{3}\,\epsilon\right) \sin 2\phi\ .\ \frac{h^2}{r}\,,$$

which is practically the same as formula (λ).

which is just the formula which would be obtained by substituting in my formula (κ) the expressions for $-\partial g/\partial\xi$ which is given by (0_2).

Thus it is clear that Dr. WOODWARD's own methods when executed with sufficient precision lead to my results, or rather to those special cases of my results which correspond to his special assumptions. From these results we conclude:

(1) That for Dr. WOODWARD's potential function (46) as well as for the approximation thereto made by assuming ν_1^2 and ν_2^2 to be constants, the meridional deviation is *towards*, and *not away from* the equator.

(This follows from equations (π_1) and (π_2).)

(2) that the meridional deviation when expanded in powers of the time of fall, begins with the *fourth*, and *not with the second* power. (This follows from formula (κ), since h varies as the square of the time of fall).

(3) that for the data: $h = 49024$ cm., $\phi = 45°$, the southerly deviation for Dr. WOODWARD's potential function (46) is

$S.D. = +.021$ cm., (by formula (π_1))

and for his approximation made by assuming ν_1^2 and ν_2^2 to be constants,

$S.D. = +.0148$ cm., (by formula (π_2))

as against the value -3.03 cm. which he obtains.

(4) that for the potential function (46) as well as for the approximation thereto, already referred to, the point P_1 *is farther away from*, and *not nearer to*, the equator than the point T in which the normal at P_0 pierces the tangent plane to the equipotential surface through P_1. (This follows from the fact, well known in Geodesy, that the lines of force of the weight field are concave in that direction on a level surface in which the rate of increase of the acceleration is greatest. Since for a distribution of revolution $\partial g/\partial\eta \equiv 0$, the concavity is to the north if $-\partial g/\partial\xi > 0$, and by ($0_1$) and ($0_2$) this is the case for both potential function (46) and its approximation. See also Part II, §4, of this paper.)

In addition to the evidence which I have already given of the correctness of my results, I have, in a paper soon to be published, and abstracted in the *Bulletin of the American Mathematical Society*, Vol. XX, No. 4, p. 175, given a geometric proof of my formula (κ). Furthermore, I give in Part II of this paper a new proof of my general formulas (ν). This proof is analytic, whereas the proof given in my second paper was somewhat geometric.

ADDENDUM.

Since this paper was presented for publication Professor F. R. MOULTON published in the *Annals of Mathematics* (second series, Vol. 15, No. 4, pp. 184–194) an article entitled "The Deviations of Falling Bodies." In this article Professor MOULTON treats the problem which was treated by Dr. WOODWARD in the article reviewed above. Like Dr. WOODWARD, he confines his treatment to the case of a distribution of revolution and to the particular distribution which corresponds to the potential function used by Dr. WOODWARD. His results agree, however, with those which I obtain (in my first paper) for the potential function which he uses. This, he himself states at the bottom of page 184 of his article.

For the southerly deviation Professor MOULTON derives the formula (his equation 35)

$$\varphi_1 - \varphi_2 = \left[\frac{1}{2}\,\frac{r_0^3}{\alpha}\left(\frac{5\beta}{r_0^5} - 4\omega^2\right)\sin 2\,\varphi_0 + \cdots\right]h^2 + \cdots,$$

in which φ_0, φ_1, φ_2 are the geocentric latitudes respectively of the initial point P_0, the point P_1 at which the falling particle meets the level surface on which the deviations are measured, the point P_2 which is the foot of the perpendicular from P_0 to the level surface of P_1, and $r_0(1+h)$ and r_0 are the distances from the center O of the Earth to the points P_0 and $P_0^{(0)}$ respectively, the latter point being the intersection of the level surface of P_1 by the radius vector to P_0. These coördinates are represented in Fig. 4. The quantity ω is the value of the angular velocity of the Earth's rotation and the quantities α and β are given by the formulas:

$$\alpha = \kappa M\ ,\ \beta = \frac{\kappa}{2}\,(C-A)\ ,$$

where M is the mass of the Earth (in c. g. s. units), κ the gravitation constant, and C and A are the principal moments of inertia of the Earth, C being that with respect to the axis of rotation and A that with respect to an equatorial axis.

From these values of α and β it follows that

$$4\omega^2\frac{r_0^3}{\alpha} = 4\,\frac{\omega^2 r_0}{\kappa M/r_0^2}\ ,\quad \frac{5\beta}{r_0^5}\,\frac{r_0^3}{\alpha} = \frac{5\beta}{\alpha r_0^2} = \frac{5}{2}\,\frac{C-A}{M\,r_0^2}$$

By the relations which are given between equations (40) and (41) which follow, these expressions are practically equal to $4\sigma_1$ and $\frac{5}{3}\epsilon$, respectively, where σ_1 and ϵ have the values given by the equations (μ) above.

Hence Professor MOULTON's formula may be written as follows:—

$$\varphi_2 - \varphi_1 = \tfrac{1}{2}\,(4\sigma_1 - \tfrac{5}{3}\epsilon)\sin 2\varphi_0 \cdot h^2\ .$$

Since Professor MOULTON's symbols φ_0, r_0 $(\varphi_2-\varphi_1)$, r_0h stand for the same quantities respectively as my symbols ψ_0, $S.D.$, h, it follows that Professor MOULTON's formula is the same as my formula (λ) above.

Professor MOULTON also states on page 194 of his article that "the deviation of a body freely falling a small distance near the Earth's surface is equatorward for all latitudes between 0 and $\pm 90°$," meaning, of course, for

the particular potential function which he uses, which function is the same as Dr. WOODWARD's function (46) and function (38) of this paper. This statement agrees with conclusion (4) which is stated in this paper below equations (π_2) above.

§1. RESULT OF DR. WOODWARD'S FIRST METHOD WHEN CARRIED THROUGH WITHOUT MAKING APPROXIMATIONS.

Dr. WOODWARD's first method is given in the section of his paper (*A.J.* Nos. 651-652) headed: "Integration of Equations (24)."

Instead of using Dr. WOODWARD's approximate relations (18) and (19), we will use his exact relations (8), which we will put in a different form by means of his exact relations (7), (12), (13), (14).

By the first of (14) and the last of (13),

$$\theta = \varphi - \psi_1 = \delta + \delta_0$$

Hence

$$\cos(\varphi - \psi_1) = \cos(\delta + \delta_0) = \cos\delta\cos\delta_0 - \sin\delta\sin\delta_0\,,$$
$$\sin(\varphi - \psi_1) = \sin(\delta + \delta_0) = \sin\delta\cos\delta_0 + \cos\delta\sin\delta_0\,.$$

By (13)

$$r_1\cos\delta = r_0 - h\cos\delta_0 \quad , \quad r_1\sin\delta = h\sin\delta_0\,,$$

and hence

$$-r_1\cos(\varphi - \psi_1) = h - r_0\cos\delta_0\,,$$
$$-r_1\sin(\varphi - \psi_1) = \quad - r_0\sin\delta_0\,.$$

Also by (7) and (12),

$$\varphi - \psi = \varphi - \psi_0 + \Delta\psi = \delta_0 + \Delta\psi\,.$$

Hence

$$\cos(\varphi - \psi) = \cos(\delta_0 + \Delta\psi) = \cos\delta_0\cos\Delta\psi - \sin\delta_0\sin\Delta\psi\,,$$
$$\sin(\varphi - \psi) = \sin(\delta_0 + \Delta\psi) = \sin\delta_0\cos\Delta\psi + \cos\delta_0\sin\Delta\psi\,.$$

Since

$$\sin\Delta\psi = \Delta\psi - \tfrac{1}{6}(\Delta\psi)^3 + \ldots\,,\quad \cos\Delta\psi = 1 - \tfrac{1}{2}(\Delta\psi)^2 + \ldots\,,$$

it follows that

$$r\cos(\varphi - \psi) = r\cos\delta_0 - r\Delta\psi\sin\delta_0 - \tfrac{1}{2}r(\Delta\psi)^2\cos\delta_0 + \ldots\,,$$
$$r\sin(\varphi - \psi) = r\sin\delta_0 + r\Delta\psi\cos\delta_0 - \tfrac{1}{2}r(\Delta\psi)^2\sin\delta_0 + \ldots\,,$$

Therefore the third and the first of Dr. WOODWARD's equations (8) may be written as follows:

(I)

$$\xi = h + (r - r_0)\cos\delta_0 - r\Delta\psi\sin\delta_0 - \tfrac{1}{2}r(\Delta\psi)^2\cos\delta_0$$
$$+ \ldots - 2r\cos\psi\cos\varphi\sin^2\tfrac{1}{2}\Delta\lambda\,,$$

$$\xi = (r - r_0)\sin\delta_0 + r\Delta\psi\cos\delta_0 - \tfrac{1}{2}r(\Delta\psi)^2\sin\delta_0 + \ldots$$
$$- 2r\cos\psi\sin\varphi\sin^2\tfrac{1}{2}\Delta\lambda\,.$$

In order to integrate Dr. WOODWARD's equations (24) let us transform them by the transformation:

$$(i) \qquad \tau = r\cos\psi \quad , \quad \sigma = r\sin\psi \quad , \quad \lambda = \lambda\,,$$

and thus obtain the equations: *(ii)*

$$\tau'' - \tau(\omega + \lambda')^2 = \frac{\partial V}{\partial\tau}\,,\quad \sigma'' = \frac{\partial V}{\partial\sigma}\,,\quad \frac{d}{dt}\Big[\tau^2(\omega + \lambda')\Big] = \frac{\partial V}{\partial\lambda}$$

Since $\partial V / \partial\lambda = 0$, the last equation yields the first integral

$$\tau^2(\omega + \lambda') = \text{constant} = \tau_0^2\omega\,,$$

whence

$$\omega + \lambda' = \frac{\tau_0^2}{\tau^2}\,\omega\,, \tag{iii}$$

and therefore the first two equations become*

$$\tau'' = \frac{\partial V}{\partial\tau} + \frac{\tau_0^4}{\tau^3}\,\omega^2 = \frac{\partial\Omega}{\partial\tau}\,,\quad \sigma'' = \frac{\partial V}{\partial\sigma} = \frac{\partial\Omega}{\partial\sigma}\,, \tag{iv}$$

where

$$\Omega = V - \frac{\omega^2\tau_0^4}{2\tau^2}\,. \tag{v}$$

The solution of these equations, for the initial conditions:

when $\quad t = 0\,,\quad \tau = \tau_0\,,\quad \sigma = \sigma_0\,,\quad \tau' = 0\,,\quad \sigma' = 0\,.$

is

$$\tau = \tau_0 + \alpha_2 t^2 + \alpha_4 t^4 + \ldots\,, \tag{vi}$$
$$\sigma = \sigma_0 + \beta_2 t^2 + \beta_4 t^4 + \ldots\,,$$

where

$$\alpha_2 = \frac{1}{2}\,\Omega_\tau^0\,,\quad \beta_2 = \frac{1}{2}\,\Omega_\sigma^0\,,$$

$$\alpha_4 = \frac{1}{2\cdot 4!}\left[\frac{\partial}{\partial\tau}\left(\Omega_\tau^2 + \Omega_\sigma^2\right)\right]^0,\quad \beta_4 = \frac{1}{2\cdot 4!}\left[\frac{\partial}{\partial\sigma}\left(\Omega_\tau^2 + \Omega_\sigma^2\right)\right]^0,$$

the upper ^{0} denoting the particular values of the derivatives of Ω to which they are attached, for the values $\tau = \tau_0\,,\ \sigma = \sigma_0$.

Hence

$$\tau^2 + \sigma^2 = A + Bt^2 + Ct^4 + \ldots$$

and

$$r = \sqrt{\tau^2 + \sigma^2} = \sqrt{A} + \frac{B}{2\sqrt{A}}\,t^2 + \frac{4AC - B^2}{8A\sqrt{A}}\,t^4 + \ldots$$

where

$$A = \tau_0^2 + \sigma_0^2 = r_0^2\,,\quad B = 2\,(\alpha_2\tau_0 + \beta_2\sigma_0)\,,$$
$$C = \alpha_2^2 + \beta_2^2 + 2\,(\alpha_4\tau_0 + \beta_4\sigma_0)$$

and therefore

$$r = r_0 + \frac{\alpha_2\tau_0 + \beta_2\sigma_0}{r_0}\,t^2 \tag{vii}$$
$$+ \frac{(\alpha_2\sigma_0 - \beta_2\tau_0)^2 + 2r_0^2(\alpha_4\tau_0 + \beta_4\sigma_0)}{2r_0^3}\,t^4$$

also

$$\tan\psi = \frac{\sigma}{\tau} = D + Et^2 + Ft^4 + \ldots\,.$$

*These are the equations at the top of page 348 of my first paper (*T.A.M.S.*, Vol. XII), where r, z, V, f_1 stand for τ, σ, Ω, V respectively.

where

$$D = \frac{\sigma_0}{\tau_0} = \tan\psi_0 \ , \qquad E = \frac{\tau_0\beta_2 - \sigma_0\alpha_2}{\tau_0^2} \ ,$$

$$F = \frac{\tau_0(\tau_0\beta_4 - \sigma_0\alpha_4) - \alpha_2(\tau_0\beta_2 - \sigma_0\alpha_2)}{\tau_0^3} \ ,$$

and therefore

$$\tan\psi - \tan\psi_0 = Et^2 + Ft^4 + \cdots .$$

But

$$\tan\psi - \tan\psi_0 = \sec^2\psi_0 \cdot (\psi - \psi_0) + \sec^2\psi_0 \tan\psi_0 \ (\psi - \psi_0)^2 + \cdots ,$$

whence by inversion

$$\psi - \psi_0 = \cos^2\psi_0 \cdot [\tan\psi - \tan\psi_0] - \cos^4\psi_0 \tan\psi_0[\tan\psi - \tan\psi_0]^2 + \cdots ,$$

$$= \cos^2\psi_0 \cdot E \cdot t^2 + [\cos^2\psi_0 \cdot F - \cos^4\psi_0 \cdot \tan\psi_0 \cdot E^2]t^4 + \cdots .$$

Or

(*viii*)

$$\Delta\psi = \psi_0 - \psi = \frac{\sigma_0\alpha_2 - \tau_0\beta_2}{r_0^2}t^2 + \left\{ \frac{(\sigma_0\alpha_2 - \tau_0\beta_2)^2}{r_0^4}\frac{\sigma_0}{\tau_0} + \frac{\tau_0\beta_2 - \sigma_0\alpha_2}{r_0^2}\cdot\frac{\alpha_2}{\tau_0} + \frac{\sigma_0\alpha_4 - \tau_0\beta_4}{r_0^2} \right\} t^4 + \cdots .$$

Finally, by (*iii*)

$$\lambda' = \frac{\tau_0^2 - \tau^2}{\tau^2}\omega = \frac{-2\alpha_2\tau_0 t^2 - (\alpha_2^2 + 2\alpha_4\tau_0)t^4 + \cdots}{\tau_0^2 + 2\alpha_2\tau_0 t^2 + \cdots}\omega \ .$$

$$= -\frac{2\alpha_2\omega}{\tau_0}t^2 + \cdots ,$$

whence

$$\Delta\lambda = \lambda - \lambda_0 = -\frac{2}{3}\frac{\alpha_2\omega}{\tau_0}t^3 + \cdots \qquad (ix)$$

Equations (*vii*), (*viii*), (*ix*), where $\alpha_2, \beta_2, \alpha_4, \beta_4$ have the values given below equations (*vi*), are the solutions of Dr. WOODWARD's equations (24).

Let us now put these expressions for r, ψ, λ in Dr. WOODWARD's equations (8), using for the middle one the form which he gives and for the first and third the forms given in equations (I) above.

Then

$$\eta = r\cos\psi\sin\Delta\lambda = \tau[\Delta\lambda - \tfrac{1}{6}(\Delta\lambda)^3 + \cdots] ,$$

$$= (\tau_0 + \alpha_2 t^2 + \cdots)\left[-\frac{2}{3}\frac{\alpha_2\omega}{\tau_0^2}t^3 + \cdots\right] \qquad (x)$$

$$= -\tfrac{2}{3}\alpha_2 \cdot \omega \cdot t^3 + \cdots$$

Also (*xi*)

$$\zeta = h + (r - r_0)\cos\delta_0 - r\Delta\psi\sin\delta_0 + \cdots$$

$$= h + \left\{\frac{\alpha_2\tau_0 + \beta_2\sigma_0}{r_0}\cos\delta_0 - \frac{\alpha_2\sigma_0 - \beta_2\tau_0}{r_0}\sin\delta_0\right\}t^2 + \cdots$$

and

$$\xi = (r - r_0)\sin\delta_0 + r\Delta\psi\cos\delta_0 - \tfrac{1}{2}r(\Delta\psi)^2\sin\delta_0 + \cdots$$

$$= \frac{\alpha_2\tau_0 + \beta_2\sigma_0}{r_0}\sin\delta_0 \ t^2 + \Bigg\{\frac{(\alpha_2\sigma_0 - \beta_2\tau_0)^2}{2r_0^3}\sin\delta_0 + \frac{\alpha_4\tau_0 + \beta_4\sigma_0}{r_0}\sin\delta_0$$
$$+ \frac{\alpha_2\sigma_0 - \beta_2\tau_0}{r_0}\cos\delta_0 \qquad + A'\cos\delta_0 + \frac{\alpha_4\sigma_0 - \beta_4\tau_0}{r_0}\cos\delta_0$$
$$- \frac{1}{2}\frac{(\alpha_2\sigma_0 - \beta_2\tau_0)^2}{r_0^3}\sin\delta_0\Bigg\}t^4 .$$

Where

$$A' = \frac{(\alpha_2\sigma_0 - \beta_2\tau_0)^2}{r_0^3}\frac{\sigma_0}{\tau_0} + \frac{\beta_2\tau_0 - \alpha_2\sigma_0}{r_0}\frac{\alpha_2}{\tau_0} + \frac{(\alpha_2\sigma_0 - \beta_2\tau_0)(\alpha_2\tau_0 + \beta_2\sigma_0)}{r_0^3}$$

$$= \frac{(\alpha_2\sigma_0 - \beta_2\tau_0)\left[(\alpha_2\sigma_0 - \beta_2\tau_0)\sigma_0 + (\alpha_2\tau_0 + \beta_2\sigma_0)\tau_0\right] + r_0^2\alpha_2(\beta_2\tau_0 - \alpha_2\sigma_0)}{r_0^3\tau_0} = 0$$

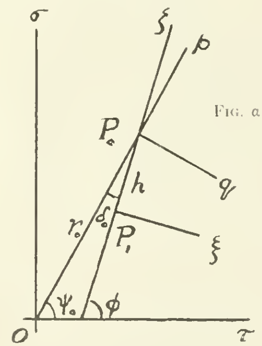

FIG. a

Therefore

$$(xii) \quad \xi = \left\{\frac{\alpha_2\tau_0 + \beta_2\sigma_0}{r_0}\sin\delta_0 + \frac{\alpha_2\sigma_0 - \beta_2\tau_0}{r_0}\cos\delta_0\right\}t^2 + \left\{\frac{\alpha_4\tau_0 + \beta_4\sigma_0}{r_0}\sin\delta_0 + \frac{\alpha_4\sigma_0 - \beta_4\tau_0}{r_0}\cos\delta_0\right\}t^4 + \cdots$$

Now let us express the coefficients of the powers of t in (*x*), (*xi*), (*xii*), in terms of the values of the derivatives of Ω at P_0 (τ_0, σ_0). For this purpose let us take a set of rectangular axes $P_0 - p, q$ in the meridian plane of P_0, and such that the negative p-axis passes through the origin O (see Fig. *a*):

Then

$$q = (\tau - \tau_0)\sin\psi_0 - (\sigma - \sigma_0)\cos\psi_0 \ ,$$
$$p = (\tau - \tau_0)\cos\psi_0 + (\sigma - \sigma_0)\sin\psi_0 \ ,$$

or

$$q = \xi\cos\delta_0 - (\zeta - h)\sin\delta_0 \ ,$$
$$p = \xi\sin\delta_0 + (\zeta - h)\cos\delta_0 \ . \qquad (xiii)$$

Hence

$$\xi = (\tau - \tau_1)\sin\phi - (\sigma - \sigma_1)\cos\phi \ ,$$
$$\zeta = (\tau - \tau_1)\cos\phi + (\sigma - \sigma_1)\sin\phi \ ,$$

since

$$\tau_0 - \tau_1 = h\cos\phi \ , \qquad \sigma_0 - \sigma_1 = h\sin\phi \ , \qquad (xiv)$$

and

$$\phi = \psi_0 + \delta_0 \ .$$

By the first of these equations of transformation ($xiii$),

$$\frac{\alpha_2\tau_0 + \beta_2\sigma_0}{r_0} = \alpha_2 \cos\psi_0 + \beta_2 \sin\psi_0 = \frac{1}{2}\Big[\Omega_\tau^{\,0}\cos\psi_0 + \Omega_\sigma^{\,0}\sin\psi_0\Big] = \frac{1}{2}\Omega_p^{\,0} ,$$

$$\frac{\alpha_2\sigma_0 - \beta_2\tau_0}{r_0} = \alpha_2 \sin\psi_0 - \beta_2 \cos\psi_0 = \frac{1}{2}\Big[\Omega_\tau^{\,0}\sin\psi_0 - \Omega_\sigma^{\,0}\cos\psi_0\Big] = \frac{1}{2}\Omega_q^{\,0} ,$$

$$\frac{\alpha_4\tau_0 + \beta_4\sigma_0}{r_0} = \frac{1}{2\cdot 4\,!}\left[\frac{\partial}{\partial p}(\Omega_\tau^2 + \Omega_\sigma^2)\right]^0 , \qquad \frac{\alpha_4\sigma_0 - \beta_4\tau_0}{r_0} = \frac{1}{2\cdot 4\,!}\left[\frac{\partial}{\partial q}(\Omega_\tau^2 + \Omega_\sigma^2)\right]^0 .$$

By the second of the equations of transformation ($xiii$),

$$\frac{\alpha_2\tau_0 + \beta_2\sigma_0}{r_0}\cos\delta_0 - \frac{\alpha_2\sigma_0 - \beta_2\tau_0}{r_0}\sin\delta_0 = \frac{1}{2}\Big[\Omega_p^{\,0}\cos\delta_0 - \Omega_q^{\,0}\sin\delta_0\Big] = \frac{1}{2}\Omega_\zeta^{\,0} ,$$

$$\frac{\alpha_2\tau_0 + \beta_2\sigma_0}{r_0}\sin\delta_0 + \frac{\alpha_2\sigma_0 - \beta_2\tau_0}{r_0}\cos\delta_0 = \frac{1}{2}\Big[\Omega_p^{\,0}\sin\delta_0 + \Omega_q^{\,0}\cos\delta_0\Big] = \frac{1}{2}\Omega_\xi^{\,0} ,$$

$$\frac{\alpha_4\tau_0 + \beta_4\sigma_0}{r_0}\sin\delta_0 + \frac{\alpha_4\sigma_0 - \beta_4\tau_0}{r_0}\cos\delta_0 = \frac{1}{2\cdot 4\,!}\left[\frac{\partial}{\partial\xi}(\Omega_\tau^2 + \Omega_\sigma^2)\right]^0 .$$

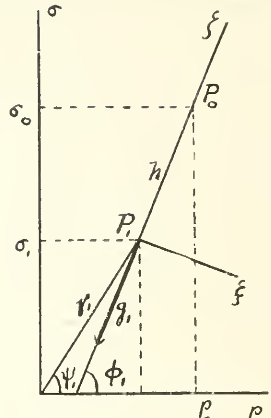

Fig. β

Therefore equations (x), (xi), (xii) may be written as follows :

$$(xv)\quad \begin{cases} \eta = -\dfrac{1}{3}\omega\,\Omega_\zeta^{\,0}\,t^3 + \cdots , \\[2mm] \zeta = h + \dfrac{1}{2}\Omega_\zeta^{\,0}\,t^2 + \cdots , \\[2mm] \xi = \dfrac{1}{2}\Omega_\xi^{\,0}\,t^2 + \dfrac{1}{2\cdot 4\,!}\left[\dfrac{\partial}{\partial\xi}(\Omega_\tau^2 + \Omega_\sigma^2)\right]^0 t^4 + \cdots . \end{cases}$$

By the last of the equations of transformation ($xiii$) the coefficient of t^4 in the last equation may be written as follows :

(xvi)

$$\frac{1}{2\,.\,4\,!}\left[\frac{\partial}{\partial\tau}(\Omega_\tau^2 + \Omega_\sigma^2)\sin\phi - \frac{\partial}{\partial\sigma}(\Omega_\tau^2 + \Omega_\sigma^2)\cos\phi\right]^0$$

$$= \frac{1}{4\,!}\Big[(\Omega_\tau^{\,0}\Omega_{\tau\tau}^{\,0} + \Omega_\sigma^{\,0}\Omega_{\sigma\tau}^{\,0})\sin\phi - (\Omega_\tau^{\,0}\Omega_{\tau\sigma}^{\,0} + \Omega_\sigma^{\,0}\Omega_{\sigma\sigma}^{\,0})\cos\phi\Big]$$

The potential function Ω was expressed in terms of the potential function V by means of relation (v). Since the direction of the vertical $P_1\,P_0$ of P_1 is determined by the weight field of force, it is desirable to introduce the potential function of that field of force. If we denote this potential function by W, we have

$$(xvii)\qquad W = V + \frac{\omega^2}{2}\tau^2 ,$$

and hence by (v)

$$(xviii)\qquad \Omega = W - \frac{\omega^2}{2}\left(\tau^2 + \frac{\tau_0^{\,4}}{\tau^2}\right) .$$

Therefore

$$\Omega_\tau^{\,0} = W_\tau^{\,0} , \quad \Omega_\sigma^{\,0} = W_\sigma^{\,0} , \quad \Omega_{\tau\tau}^{\,0} = W_{\tau\tau}^{\,0} - 4\omega^2 ,$$
$$\Omega_{\sigma\tau}^{\,0} = W_{\sigma\tau}^{\,0} , \; \Omega_{\sigma\sigma}^{\,0} = W_{\sigma\sigma}^{\,0} ,$$

and expression (xvi) may be written as follows :

$$\frac{1}{4\,!}\Big[(W_\tau^{\,0}W_{\tau\tau}^{\,0} + W_\sigma^{\,0}W_{\sigma\tau}^{\,0})\sin\phi - (W_\tau^{\,0}W_{\tau\sigma}^{\,0} + W_\sigma^{\,0}W_{\sigma\sigma}^{\,0})\cos\phi - 4\omega^2\sin\phi\,W_\tau^{\,0}\Big]$$

$$= \frac{1}{2\cdot 4\,!}\left[\frac{\partial}{\partial\xi}(W_\tau^2 + W_\sigma^2) - 4\omega^2\sin\phi\,.\,W_\tau\right]^0$$

But, by the last of ($xiii$)

$W_\tau^2 + W_\sigma^2 = W_\xi^2 + W_\zeta^2$ and $W_\tau = W_\xi\sin\phi + W_\zeta\cos\phi$.

Equations (xv) may now be written as follows :

$$\eta = -\frac{1}{3}\omega\Big[W_\zeta^{\,0}\cos\phi + W_\xi^{\,0}\sin\phi\Big]t^3 + \ldots ,$$

$$\xi = \frac{1}{2}W_\xi^{\,0}\,t^2 + \frac{1}{24}\Big[W_\xi^{\,0}W_{\xi\xi}^{\,0} + W_\zeta^{\,0}W_{\zeta\xi}^{\,0} \qquad (xix)$$
$$\qquad - 4\omega^2\sin\phi\,(W_\zeta^{\,0}\cos\phi + W_\xi^{\,0}\sin\phi)\Big]t^4 ,$$

$$\zeta = h + \frac{1}{2}W_\zeta^{\,0}\,t^2 + \ldots .$$

Since $W_\eta = 0$, W_ξ, W_ζ are the components of the vector representing the acceleration due to weight, and since the line $P_1\,P_0$ is the vertical of P_1, it follows that

$$W_\xi^{(1)} = 0 , \quad W_\zeta^{(1)} = -\,g_1 .$$

Since further, P_0 is on the vertical of P_1, and at distance h above P_1,

$$W_\xi^{\,0} = W_{\xi\zeta}^{(1)}\,h \quad , \quad W_\zeta^{\,0} = W_\zeta^{(1)} + W_{\zeta\zeta}^{(1)}\cdot h^* .$$

Hence the first two of equations (xix) may be written as follows :

* For a more detailed treatment of this sort see Part II, § 3, equations (20).

$$
(xx)\quad\left\{\begin{aligned}
\eta &= -\frac{1}{3}\omega\Big[\cos\phi\;W_\zeta^{(1)} + (\cos\phi\cdot W_{\zeta\zeta}^{(1)} \\
&\qquad + \sin\phi\;W_{\zeta\xi}^{(1)})\,h + \cdots\Big]t^3 + \cdots , \\
\xi &= \frac{1}{2}W_\xi^{(1)}\cdot h\cdot t^2 + \frac{1}{24}\Big[W_\zeta^{(1)}\,W_{\xi\zeta}^{(1)} \\
&\qquad - 4\,\omega^2\sin\phi\cos\phi\;W_\zeta^{(1)} + (\quad)\,h + \Big]t^4 + \cdots .
\end{aligned}\right.
$$

Let us now denote by $\bar{t}$ *the time of fall, i.e.*, the value of t for which $\zeta = 0$. Let the corresponding values of η and ξ be denoted by $\bar{\eta}$ and $\bar{\xi}$ respectively. Then $\bar{\eta}$ and $\bar{\xi}$ are the easterly and southerly deviations of the falling particle. In order to express these in terms of $\bar{t}$ we proceed as follows:

From the last of (xix)

$$
h = -\frac{1}{2}W_\zeta^{(1)}\;t^2 + \cdots
$$

and for this value of h, equations (xx) become

$$
(xxi)\quad\left\{\begin{aligned}
\bar{\eta} &= -\frac{1}{3}\omega\cos\phi\;W_\zeta^{(1)}\cdot\bar{t}^3 + \cdots . \\
\bar{\xi} &= -\frac{1}{24}W_\zeta^{(1)}\Big[5W_\xi^{(1)} + 4\omega^2\sin\phi\cos\phi\Big]\bar{t}^4 + \cdots
\end{aligned}\right.
$$

If further we substitute for t its value in terms of h, we get

$$
(xxii)\quad\left\{\begin{aligned}
\bar{\eta} &= \frac{2}{3}\sqrt{2}\,\omega\cos\phi\;\frac{h^{3/2}}{\sqrt{-W_\zeta^{(1)}}} + \cdots \\
\bar{\xi} &= \frac{1}{6}\Big[5W_\xi^{(1)} + 4\omega^2\sin\phi\cos\phi\Big]\frac{h^2}{-W_\zeta^{(1)}} .
\end{aligned}\right.
$$

or since

$$
-W_\zeta^{(1)} = g_1 , \qquad W_\xi^{(1)} = -(\partial g/\partial\xi)_1 .
$$

we have

$$
(xxiii)\quad\left\{\begin{aligned}
\bar{\eta} &= \frac{2}{3}\sqrt{2}\,\omega\cos\phi\;\frac{h^{3/2}}{g_1^{1/2}} + \cdots . \\
\bar{\xi} &= \frac{1}{6}\Big[4\,\omega^2\sin\phi\cos\phi - 5(\partial g/\partial\xi)_1\Big]\frac{h^2}{g_1} + \cdots .
\end{aligned}\right.
$$

These expressions for $\bar{\eta}$ and $\bar{\xi}$ are respectively the first terms of the expressions for *E. D.* and *S. D.* given by formulas (ν) above. Thus it has been shown, as stated in the introduction, that Dr. Woodward's first method when executed with sufficient precision leads to my results.

§2. Dr. Woodward's Second Method. The Effects of Some of His Assumptions.

Dr. Woodward's second method is given under the heading: "Integration of Equations (25) and (25₁)." (*A.J.*, Nos. 651-652).

We will show first that by his assumption that two quantities (denoted by ν_1^2 and ν_2^2) are constants, Dr. Woodward approximates his potential function (46) by a potential function for which the value of the acceleration g, due to weight, does not differ much from that given by (46), but for which the value of the derivative $(-\partial g/\partial\xi)$, which enters into the expression for the southerly deviation, is only $\frac{3}{5}$ as large as that corresponding to his potential function (46).

Dr. Woodward's equations (45) may be written as follows:

$$
\begin{aligned}
&\rho'' - 2\omega\eta' - \left(\omega^2 + \frac{1}{\rho}\frac{\partial V}{\partial\rho}\right)\rho = 0 , \\
&\eta'' + 2\omega\rho' - \left(\omega^2 + \frac{1}{\eta}\frac{\partial V}{\partial\eta}\right)\eta = 0 , \qquad (a) \\
&\sigma'' - \frac{1}{\sigma}\frac{\partial V}{\partial\sigma}\cdot\sigma = 0 .
\end{aligned}
$$

Since his potential function (46) is a function of $\tau = \sqrt{\rho^2 + \eta^2}$ and σ, it follows that

$$
\frac{1}{\rho}\frac{\partial V}{\partial\rho} = \frac{1}{\eta}\frac{\partial V}{\partial\eta} = \frac{1}{\tau}\frac{\partial V}{\partial\tau} . \qquad (b)
$$

By putting

$$
-\nu_1^2 = \omega^2 + \frac{1}{\tau}\frac{\partial V}{\partial\tau} \quad\text{and}\quad -\nu_2^2 = \frac{1}{\sigma}\frac{\partial V}{\partial\sigma} , \qquad (c)
$$

equations (a) become

$$
\begin{gathered}
\rho'' - 2\omega\eta' + \nu_1^2\rho = 0 , \quad \eta'' + 2\omega\,\rho' + \nu_1^2\eta = 0 . \qquad (d) \\
\sigma'' + \nu_2^2\sigma = 0 .
\end{gathered}
$$

and these are Dr. Woodward's equations (51).

In order to integrate these equations, Dr. Woodward assumes that ν_1^2 and ν_2^2 are constant quantities.

By this assumption, V is defined to be the function

$$
V = -\frac{1}{2}\Big[(\nu_1^2 + \omega^2)\tau^2 + \nu_2^2\sigma^2\Big] + \text{constant}, \qquad (e)
$$

and $W = V + \omega^2\tau^2/2$ to be the function

$$
W = V + \frac{\omega^2}{2}\tau^2 = -\frac{1}{2}\Big[\nu_1^2\tau^2 + \nu_2^2\sigma^2\Big] + \text{constant}. \qquad (f)
$$

This follows from equations (c).

The derivatives of the first order of the latter function are

$$
\frac{\partial W}{\partial\tau} = -\nu_1^2\tau , \quad \frac{\partial W}{\partial\sigma} = -\nu_2^2\sigma ,
$$

and hence the value of the acceleration due to weight is

$$
g = \sqrt{\nu_1^4\tau^2 + \nu_2^4\sigma^2} \qquad (g)
$$

at a general point (τ,σ). For the potential function (f), every level surface

$$
\tfrac{1}{2}\,\nu_1^2\tau^2 + \tfrac{1}{2}\,\nu_2^2\sigma^2 = C \qquad (h)
$$

is an ellipsoid of revolution. The value of C for the standard spheroid of major and minor semi-axes a and b, is determined by either of the relations

$$(i) \qquad a^2 = 2C/\nu_1^2 \quad , \quad b^2 = 2C/\nu_2^2 \; .$$

The equation of this standard spheroid may be written in the parametric form

$$(j) \qquad \tau = a\cos\psi \quad , \quad \sigma = b\sin\psi \quad ,$$

where ψ denotes geocentric latitude. The value of g on this spheroid is then found, by (g), (j), (i), to be,

$$\begin{aligned} g &= \sqrt{2C}\sqrt{\nu_1^2\cos^2\psi + \nu_2^2\sin^2\psi} \\ &= \sqrt{2C\nu_1^2}\sqrt{1 + \frac{\nu_2^2 - \nu_1^2}{\nu_1^2}\sin^2\psi} \\ &= \sqrt{2C\nu_1^2}\left[1 + \frac{\nu_2^2 - \nu_1^2}{2\nu_1^2}\sin^2\psi + \cdots\right]. \end{aligned}$$

If we denote by g^a the value of g at the equator of the standard spheroid, we have by equations (g) and (i),

$$g_a = \nu_1^2 a \quad , \quad \sqrt{2C\nu_1^2} = g_a \quad ,$$

and hence

$$(k) \qquad g = g_a\left[1 + \frac{a}{g_a}\left(\frac{\nu_2^2 - \nu_1^2}{2}\right)\sin^2\psi + \cdots\right].$$

If further, we denote by ξ distance, measured to the south, along a meridian of the standard spheroid, we have approximately

$$-d\xi = a\,d\psi \quad ,$$

and hence by differentiation of equation (k),

$$(l) \qquad -\frac{\partial g}{\partial\xi} = \frac{\nu_2^2 - \nu_1^2}{2}\sin 2\psi \; .$$

The value of the derivative $-\partial g/\partial\xi$ may be gotten more accurately as follows. For the potential function (f), which is of the form $W = f(\tau, \sigma)$, we have

$$(m) \qquad \frac{\partial^2 f}{\partial\tau^2} = -\nu_1^2 \quad , \quad \frac{\partial^2 f}{\partial\tau\partial\sigma} = 0 \quad , \quad \frac{\partial^2 f}{\partial\sigma^2} = -\nu_2^2 \quad ,$$

and therefore by the last relation of the following footnote*,

$$(n) \qquad -\frac{\partial g}{\partial\xi} = \frac{\nu_2^2 - \nu_1^2}{2}\sin 2\phi \quad ,$$

where ϕ denotes the astronomical latitude.

According to Dr. WOODWARD (*A.J.* Nos. 651-652, p. 26, second column)

$$\nu_2^2 = 15449.27 \times 10^{-10} \quad , \quad \nu_1^2 = 15346.62 \times 10^{-10} \quad .$$

Hence

$$(o) \qquad \left.\begin{aligned} \frac{a}{g_a}\,\frac{\nu_2^2 - \nu_1^2}{2} &= \frac{\nu_2^2 - \nu_1^2}{2\nu_1^2} = .00334 \\ \text{and} \qquad \frac{\nu_2^2 - \nu_1^2}{2} &= 5.1325 \times 10^{-9} \end{aligned}\right\}$$

The first of these two numbers does not differ by more than one unit in the last decimal place, from the common oblateness $\dfrac{a-b}{a} = \dfrac{\nu_2 - \nu_1}{\nu_2} = .00335$ of the equipotential surfaces (h).

Thus we see that corresponding to the potential function (f), which is the approximation made by Dr. WOODWARD to his potential function (46) by virtue of his assumption that ν_1^2 and ν_2^2 are constants, the values of g and $\partial g/\partial\xi$ on the standard spheroid are given by the formulas:

$$(p) \qquad g = g_a\left[1 + .0033\sin^2\psi\right], \quad -\frac{\partial g}{\partial\xi} = 5.132 \times 10^{-9}\sin 2\psi,$$

where g_a is the value of g at the equator and ψ denotes geocentric latitude. In Part II (§4, eqs. 45, 46), of this paper it is shown that for the potential function which is the same as Dr. WOODWARD's function (46), the corresponding formulas for the same standard spheroid are

$$(q) \qquad g = g_a\left[1 + .0052\sin^2\psi\right], \quad -\frac{\partial g}{\partial\xi} = 8.14 \times 10^{-9}\sin 2\psi \; .$$

Since $g_a = 978.06$, the value of g given by the first of formulas (p) does not differ much from that given by the first of formulas (q). However, the value of $-\partial g/\partial\xi$ given by the second of formulas (p) is only about $5/8$ as large as that given by the second of formulas (q).

Thus we have proven the statement made at the beginning of this section, and also the first part of the statement which was made concerning this section in the introduction.

*If the potential function W of the weight field of force is of the form

$$W = f(\tau, \sigma) \qquad \text{so that} \qquad \frac{\partial W}{\partial\lambda} = 0 \; ,$$

the value of the acceleration at a general point P, (τ, σ), is

$$g = \sqrt{\left(\frac{\partial f}{\partial\tau}\right)^2 + \left(\frac{\partial f}{\partial\sigma}\right)^2} = -\left[\frac{\partial f}{\partial\tau}\cos\phi + \frac{\partial f}{\partial\sigma}\sin\phi\right],$$

where ϕ denotes the astronomical latitude of P. Also

$$\frac{\partial f}{\partial\tau}\sin\phi - \frac{\partial f}{\partial\sigma}\cos\phi = 0$$

and therefore

$$-g\cos\phi = \frac{\partial f}{\partial\tau} \; , \qquad -g\sin\phi = \frac{\partial f}{\partial\sigma} \; .$$

REMARK. If we admit (what I have proven in both of my published papers and in Part II of this paper) that the southerly deviation of a falling particle is given by the formula:

$$S.D. = \left[2\omega^2 \sin 2\phi - 5(\partial g/\partial\xi)\right]\frac{h^2}{6g},$$

where

$$\omega^2 = 5.3173 \times 10^{-9},$$

we have by (n) and the last of (o) [or by the second of (p),] *for potential function* (f),

$$(r) \qquad S.D. = \left[2\omega^2 + 5\,\frac{\nu_2^2 - \nu_1^2}{2}\right]\sin 2\phi \cdot \frac{h^2}{6g}$$
$$= 36.29 \times 10^{-9} \sin 2\phi \; \frac{h^2}{6g},$$

and by the last of (q),

for Dr. Woodward's function (46),

$$(s) \qquad S.D. = 51.33 \times 10^{-9} \sin 2\phi \; \frac{h^2}{6g}.$$

Therefore the assumption that ν_1^2 and ν_2^2 are constants reduces by about 30% the magnitude of the southerly deviation, from that corresponding to Dr. WOODWARD's potential function (46).

We will now show that if Dr. WOODWARD's exact solution (56), (57) of his differential equations (51), in which ν_1^2 and ν_2^2 are regarded as constants, be transformed to the coördinates ξ, ζ by means of his exact equations (10), there will result for the southerly deviation the formula r which is obtained from my general formula (κ) by substituting for the derivative $\partial g/\partial\xi$ the expression of this derivative which corresponds to the approximation to Dr. WOODWARD's potential function (46) made by assuming ν_1 and ν_2 to be constants.

If we expand the sine and cosine of wt and ωt in powers of t, the first of Dr. WOODWARD's equations (56) and (57) become

$$(t) \qquad \begin{aligned} \rho &= \rho_0\left\{1 - \frac{1}{2}(w^2 - \omega^2)t^2 + \frac{1}{24}(w^4 + 2w^2\omega^2 - 3\omega^4)t^4 + \cdots\right\}, \\ \sigma &= \sigma_0\left\{1 - \frac{1}{2}\nu_2^2 t^2 + \frac{1}{24}\nu_2^4 t^4 + \cdots\right\}, \end{aligned}$$

where

$$w^2 = \nu_1^2 + \omega^2.$$

From Dr. WOODWARD's exact equations of transformation* (10) we get the following equations:

$$(u) \qquad \begin{aligned} \xi &= (\rho - \rho_1)\sin\phi_1 - (\sigma - \sigma_1)\cos\phi_1, \\ \zeta &= (\rho - \rho_1)\cos\phi_1 + (\sigma - \sigma_1)\sin\phi_1, \end{aligned}$$

where

$$\rho_1 = r_1\cos\psi_1, \quad \sigma_1 = r_1\sin\psi_1.$$

NOTE. In order to distinguish the geographic latitude of P_1 (which Dr. WOODWARD denotes by ϕ) from the latitudes of other points, I will denote the latitude of P_1 by ϕ_1, thus using ϕ_1 where Dr. WOODWARD uses ϕ. By (u) and (t)

$$(v) \qquad \begin{aligned} \xi = {} & (\rho_0 - \rho_1)\sin\phi_1 - (\sigma_0 - \sigma_1)\cos\phi_1 \\ & + \frac{1}{2}\left[\nu_2^2\sigma_0\cos\phi_1 - \nu_1^2\rho_0\sin\phi_1\right]t^2 \\ & + \frac{1}{24}\left[\nu_1^4\rho_0\sin\phi_1 - \nu_2^4\sigma_0\cos\phi_1 + 4\nu_1^2\rho_0\sin\phi_1\cdot\omega^2\right]t^4 + \cdots, \\ \zeta = {} & (\rho_0 - \rho_1)\cos\phi_1 + (\sigma_0 - \sigma_1)\sin\phi_1 \\ & - \frac{1}{2}\left[\nu_2^2\sigma_0\sin\phi_1 + \nu_1^2\rho_0\cos\phi_1\right]t^2 + \cdots. \end{aligned}$$

In order to evaluate the coefficients of these series, we make use of the following relations:

$$(w) \qquad \left.\begin{aligned} \rho_0 - \rho_1 &= h\cos\phi_1, & \sigma_0 - \sigma_1 &= h\sin\phi_1, \\ \nu_1^2\rho_1 &= g_1\cos\phi_1, & \nu_2^2\sigma_1 &= g_1\sin\phi_1. \end{aligned}\right\}$$

The first two of these relations are easily deduced from Fig. β (p. 184). The other two may be obtained as follows. By Dr. WOODWARD's equations (51) (or equations (d) above), in which ν_1 and ν_2 are regarded as constants, the components of the acceleration of a particle starting from rest with respect to the axes $0 - \rho, \eta, \sigma$ (which are fixed with respect to the solid part of the rotating Earth and are shown in Fig. 3, p. 191) are

$$-\nu_1^2\rho, \quad -\nu_1^2\eta, \quad -\nu_2^2\sigma,$$

and hence the magnitude of the acceleration at a general point is

$$g = \sqrt{\nu_1^4(\rho^2 + \eta^2) + \nu_2^4\sigma^2}.$$

For the particular point P_1, $(\rho_1, 0, \sigma_1)$ the value of g is

$$g_1 = \sqrt{\nu_1^4\rho_1^2 + \nu_2^4\sigma_1^2}.$$

Furthermore

$$\begin{aligned} \frac{\partial g}{\partial\tau} &= \frac{1}{g}\left[\frac{\partial f}{\partial\tau}\cdot\frac{\partial^2 f}{\partial\tau^2} + \frac{\partial f}{\partial\sigma}\cdot\frac{\partial^2 f}{\partial\sigma\partial\tau}\right] \\ &= -\left[\frac{\partial^2 f}{\partial\tau^2}\cos\phi + \frac{\partial^2 f}{\partial\sigma\partial\tau}\sin\phi\right], \\ \frac{\partial g}{\partial\sigma} &= \frac{1}{g}\left[\frac{\partial f}{\partial\tau}\cdot\frac{\partial^2 f}{\partial\tau\partial\sigma} + \frac{\partial f}{\partial\sigma}\cdot\frac{\partial^2 f}{\partial\sigma^2}\right] \\ &= -\left[\frac{\partial^2 f}{\partial\tau\partial\sigma}\cos\phi + \frac{\partial^2 f}{\partial\sigma^2}\sin\phi\right]. \end{aligned}$$

If we denote by ξ distance measured to the south along the meridian of the level surface which passes through the general point P, we have

$$\begin{aligned} \frac{\partial g}{\partial\xi} &= \frac{\partial g}{\partial\tau}\sin\phi - \frac{\partial g}{\partial\sigma}\cos\phi \\ &= \frac{\partial^2 f}{\partial\tau\partial\sigma}\left(\cos^2\phi - \sin^2\phi\right) + \left(\frac{\partial^2 f}{\partial\sigma^2} - \frac{\partial^2 f}{\partial\tau^2}\right)\sin\phi\cos\phi \\ &= \frac{1}{g^2}\left\{\frac{\partial^2 f}{\partial\tau\partial\sigma}\left[\left(\frac{\partial f}{\partial\tau}\right)^2 - \left(\frac{\partial f}{\partial\sigma}\right)^2\right] + \left[\frac{\partial^2 f}{\partial\sigma^2} - \frac{\partial^2 f}{\partial\tau^2}\right]\frac{\partial f}{\partial\tau}\frac{\partial f}{\partial\sigma}\right\}. \end{aligned}$$

*In the first two of Dr. WOODWARD's equations (10), $\sin\psi$ and $\cos\psi$ should be interchanged.

The last two of equations (w) follow from the fact that the vector representing g_1 acts along the line $P_0 P_1$. See Fig. β, p. 184.

By the first two of equations (w),

$$(x_1) \qquad \begin{aligned} (\rho_0 - \rho_1) \sin \phi_1 - (\sigma_0 - \sigma_1) \cos \phi_1 &= 0 \; , \\ (\rho_0 - \rho_1) \cos \phi_1 + (\sigma_0 - \sigma_1) \sin \phi_1 &= h \; . \end{aligned}$$

These relations also follow from equations (u), because when $\rho = \rho_0$ and $\sigma = \sigma_0$, $\xi = 0$ and $\zeta = h$.

The co-efficients of t^2 in equations (v) may be written as follows:

$$\frac{1}{2} \left\{ \nu_2^2 \sigma_1 \cos \phi_1 - \nu_1^2 \rho_1 \sin \phi_1 + \nu_2^2 (\sigma_0 - \sigma_1) \cos \phi_1 - \nu_1^2 (\rho_0 - \rho_1) \sin \phi_1 \right\} ,$$

$$-\frac{1}{2} \left\{ \nu_2^2 \sigma_1 \sin \phi_1 + \nu_1^2 \rho_1 \cos \phi_1 + \nu_2^2 (\sigma_0 - \sigma_1) \sin \phi_1 + \nu_1^2 (\rho_0 - \rho_1) \cos \phi^1 \right\}$$

By relations (w), these may be written respectively as follows:

$$(x_2) \qquad \begin{aligned} &\frac{1}{2} h \, (\nu_2^2 - \nu_1^2) \sin \phi_1 \cos \phi_1 \; , \\ &-\frac{1}{2} g_1 - \frac{1}{2} h \, (\nu_2^2 \sin^2 \phi_1 + \nu_1^2 \cos^2 \phi_1) \end{aligned}$$

The co-efficient of t^4 in the first of equations (v) may be written as follows:

$$\frac{1}{24} \left\{ \nu_1^4 \rho_1 \sin \phi_1 - \nu_2^4 \sigma_1 \cos \phi_1 + 4 \nu_1^2 \rho_1 \sin \phi_1 \; \omega^2 + \nu_1^4 (\rho_0 - \rho_1) \sin \phi_1 - \nu_2^4 (\sigma_0 - \sigma_1) \cos \phi_1 + 4 \nu_1^2 (\rho_0 - \rho_1) \sin \phi_1 \; \omega^2 \right\} .$$

By relations (w) this may be written as follows:

$$(x_3) \qquad \frac{1}{24} \left\{ g_1 \sin \phi_1 \cos \phi_1 \; (\nu_1^2 - \nu_2^2 + 4\omega^2) + h \sin \phi_1 \cos \phi_1 \; (\nu_1^4 - \nu_2^4 + 4 \nu_1^2 \omega^2) \right\} .$$

With the values of the coefficients given in (x_1), (x_2), (x_3), equations (v) may be written as follows:

$$(y) \qquad \begin{aligned} \xi &= \frac{\nu_2^2 - \nu_1^2}{2} \cdot \frac{h}{2} \sin 2\phi_1 \; t^2 + \frac{1}{24} \left\{ g_1 \left[\frac{\nu_1^2 - \nu_2^2}{2} + 2\omega^2 \right] + h \, \frac{\nu_1^4 - \nu_2^4 + 4 \nu_1^2 \omega^2}{2} \right\} \sin 2\phi_1 \; t^4 \\ \zeta &= h - \frac{1}{2} \left\{ g_1 t^2 + h \, (\nu_2^2 \sin^2 \phi_1 + \nu_1^2 \cos^2 \phi_1) \right\} t^2 + \cdots . \end{aligned}$$

Let us now denote by $\bar{t}$ the time of fall. Then for $t = \bar{t}$, $\zeta = 0$ and $\xi = \bar{\xi}$, where $\bar{\xi}$ is the southerly deviation. Then from the last of the above equations

$$h = \frac{\frac{1}{2} g_1 \bar{t}^2 + \cdots}{1 - (\nu_2^2 \sin^2 \phi_1 + \nu_1^2 \cos^2 \phi_1) \bar{t}^2 + \cdots} = \tfrac{1}{2} g_1 \bar{t}^2 + \cdots$$

whence the first equation becomes

$$(z) \qquad \begin{aligned} \bar{\xi} &= \frac{1}{24} \left\{ 5 \, \frac{\nu_2^2 - \nu_1^2}{2} + 2\omega^2 \right\} g_1 \sin 2\phi_1 \; \bar{t}^4 + \cdots \\ &= \left\{ 5 \, \frac{\nu_2^2 - \nu_1^2}{2} + 2\omega^2 \right\} \sin 2\phi_1 \; \frac{h^2}{6 g_1} + \cdots . \end{aligned}$$

which is the same as formula (r) given above.

PART II.*

§ 1. Definitions of the Deviations.

Let us assume a set of rectangular axes $O - u, v, z$, which is at rest with respect to the solid part of the rotating earth, and such that Oz coincides with the earth's axis of rotation, the positive sense of Oz being from the celestial south pole to the celestial north pole. The field of force which determines weight, *i.e.*, the field in which a plumb-line is in equilibrium, is at rest with respect to these axes. We shall call this field of force the *weight field*.

Let us denote by P_1 a point (on or near the earth's surface) which is at rest with respect to the axes $O - u, v, z$. The straight line which passes through P_1 and gives the direction of the force of the weight field at P_1, is defined as *the vertical of* P_1. This vertical coincides with the string of a plumb-line, the bob of which is situated at P_1.† In general the vertical of a point does *not* intersect the axis of rotation Oz. *The astronomical meridian plane of* P_1 is the plane which passes through the vertical of P_1 and is parallel to the axis of rotation Oz. *The astronomical latitude of* P_1 is the complement ϕ_1 of the angle which the vertical of P_1 (to the zenith) makes with the axis of rotation Oz (to the celestial north pole.) *The astronomical longitude of* P_1 is the angle ϵ_1 which the meridian plane of P_1 makes with a fixed plane (zOu, say) through Oz, and is measured from 0° to 360° to the east. *The horizontal plane of* P_1 is the plane which passes through P_1 and is perpendicular to the vertical of P_1. *The north-and-south line of* P_1 is the line of intersection of the meridian and horizontal planes of P_1. *The-east-and-west line of* P_1 is the straight line which passes through P_1 and is perpendicular to the meridian plane of P_1. This line is the intersection of the horizontal plane of P_1 by the plane which passes through P_1 and is perpendicular to the axis of rotation Oz.‡ By *the*

*The derivation of the formulas for the deviations which is given in Part II, was presented to the American Mathematical Society at its New York meeting, Feb. 28, 1914. See Bulletin American Mathematical Society, Vol. XX, No. 8, p. 402.

†It is assumed that the string of the plumb-line is weightless and perfectly flexible, and that the bob is a heavy particle.

‡The above definitions are practically the same as those given by *Pizzetti, Trattato di Geodesia teoretica* (1905) § 5.

cardinal axes of P_1 we shall mean the rectangular axes composed of the vertical of P_1, the north-and-south line of P_1 and the east-and-west line of P_1. The positive senses of these axes shall be those to the zenith, to the south, and to the east, respectively. Fig. 1 represents in orthographic projection the point P_1

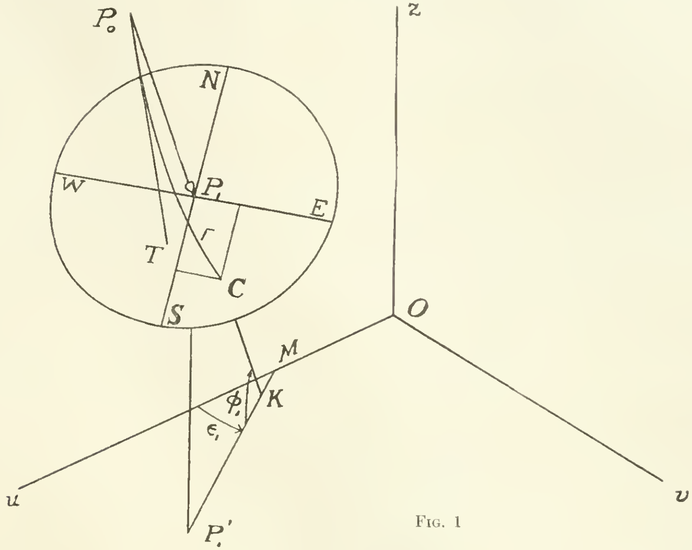

Fig. 1

(whose projection on the plane uOv is P_1'), the vertical P_1K of P_1 (whose projection on the plane uOv is $P_1'K$), the astronomical latitude $\phi_1 = \measuredangle P_1' K P_1$ of P_1, the astronomical meridian plane $K P_1 P_1'$ of P_1, the astronomical longitude $\epsilon_1 = \measuredangle u M P_1'$ of P_1, the horizontal plane $N E S W$ of P_1, the north-and-south line $N P_1 S$ of P_1, and the east-and-west line $E P_1 W$ of P_1.

Now let us take a point P_0 above P_1 and in the vertical of P_1. The vertical of P_0 will *not* coincide with the vertical $P_0 P_1$ of P_1, but it will pierce the horizontal plane $N E S W$ of P_1 in a point T which does not coincide with P_1. Let us suppose a material particle to fall, under the influence of the earth's gravitation, from an initial position of rest (with respect to the axes $O - u, v, z$) at P_0. The path of this falling particle, with respect to the axes $O - u, v, z$, is a curve l which pierces the horizontal plane $N E S W$ of P_1 in a point C.

Definition 1. *The perpendicular distance of the point* C *from the north-and-south line* $N P_1 S$ *of* P_1, *is called the easterly deviation of the falling particle, and is regarded as positive to the east.*

Definition 2. *The perpendicular distance of the point* C *from the east-and-west line* $E P_1 W$ *of* P_1, *is called the meridional or southerly deviation of the falling particle, and is regarded as positive to the south.*

These are the definitions, stated in precise geometric form, which are used by both Dr. Woodward (*A. J.*, Nos. 651-652, p. 19, first column) and myself (*T. A. M. S.*, Vol. XIII, p. 473, §1.)

§2. Equations of Motion of the Falling Particle.

In order to derive the equations of motion in the most elementary manner let us choose a set of rectangular axes $O - x, y, z$ of which the origin O and the axis Oz are the same as those of the axes $O - u, v, z$, and such that the positive part of the axis Ox is directed to the vernal equinox. See Fig. 2. The axes $O - u, v, z$ rotate around Oz, with respect to the axes $O - x, y, z$, once in a sidereal day.

Let us denote by V the potential per unit mass at a general point P exterior to the earth, by s the distance from P of an element of mass dm of the earth, and by κ the gravitation constant. Then

$$V = \kappa \int \frac{dm}{s} \,. \tag{1}$$

and the equations of motion of the falling particle are

$$\frac{d^2x}{dt^2} = \frac{\partial V}{\partial x}, \quad \frac{d^2y}{dt^2} = \frac{\partial V}{\partial y}, \quad \frac{d^2z}{dt^2} = \frac{\partial V}{\partial z} \,. \tag{2}$$

If we denote by ι the angle xOu (Fig. 2),

$$\frac{d\iota}{dt} = \omega\ ,$$

where ω is the angular velocity of the earth's rotation, and the equations of transformation from the coördinates x, y, z to the coördinates u, v, z are

$$(3)\qquad x = u\cos\iota - v\sin\iota\ ,\qquad y = u\sin\iota + v\cos\iota\ ,\qquad z = z\ .$$

By this transformation the equations (2) are transformed into the equations:

(4)

$$\frac{d^2u}{dt^2} - 2\omega\frac{dv}{dt} - \omega^2 u = \frac{\partial V}{\partial u}\,,\quad \frac{d^2v}{dt^2} + 2\omega\frac{du}{dt} - \omega^2 v = \frac{\partial V}{\partial v}\,,\quad \frac{d^2z}{dt^2} = \frac{\partial V}{\partial z}\,.$$

These are the equations of motion with respect to the axes $O - u, v, z$. They may be put into a somewhat shorter form by the introduction of the potential function of the weight field,

$$W = V + \frac{\omega^2}{2}(u^2 + v^2)\ . \qquad (5)$$

By means of relation (5) equations (4) take the form

$$\frac{d^2u}{dt^2} - 2\omega\frac{dv}{dt} = \frac{\partial W}{\partial u}\,,\quad \frac{d^2v}{dt^2} + 2\omega\frac{du}{dt} = \frac{\partial W}{\partial v}\,,\quad \frac{d^2z}{dt^2} = \frac{\partial W}{\partial z}\,. \qquad (6)$$

Since the deviations of the falling particle are referred to the cardinal axes of P_1, let us refer the motion to these axes. Let us denote by η, ξ, ζ distances measured along the east-and-west line, the north-and-south line and the vertical respectively, the positive senses being to the east, south and zenith (see Fig. 2.) The equations of transformation from the coördinates u, v, z to η, ξ, ζ are

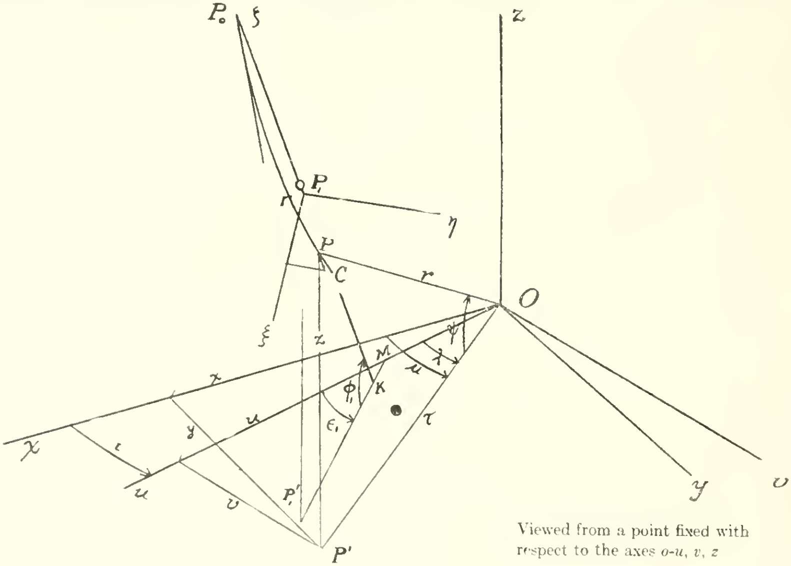

FIG. 2

$$(7)\qquad \begin{aligned} u &= u_1 + (\xi\sin\phi_1 + \zeta\cos\phi_1)\cos\epsilon_1 - \eta\sin\epsilon_1\ ,\\ v &= v_1 + (\xi\sin\phi_1 + \zeta\cos\phi_1)\sin\epsilon_1 + \eta\cos\epsilon_1\ ,\\ z &= z_1 - \xi\cos\phi_1 + \zeta\sin\phi_1\ .\end{aligned}$$

When subjected to this transformation the system of equations (6) is transformed into the system*:

$$\begin{aligned} &\frac{d^2\xi}{dt^2} - 2\omega\sin\phi_1\ \frac{d\eta}{dt} = \frac{\partial W}{\partial \xi}\,,\\ &\frac{d^2\eta}{dt^2} + 2\omega\left(\sin\phi_1\,.\,\frac{d\xi}{dt} + \cos\phi_1\,.\,\frac{d\zeta}{dt}\right) = \frac{\partial W}{\partial \eta}\,. \qquad (8)\\ &\frac{d^2\zeta}{dt^2} - 2\omega\cos\phi_1\,.\,\frac{d\eta}{dt} = \frac{\partial W}{d\zeta}\,.\end{aligned}$$

*If we had subjected the equations (4) to the transformation (7) we would have obtained the equations

(9)

$$\xi'' - 2\omega\sin\phi_1\,.\,\eta' - \omega^2\sin\phi_1\,[\xi\sin\phi_1 + \zeta\cos\phi_1 + u_1\cos\epsilon_1 + v_1\sin\epsilon_1] = \frac{\partial V}{\partial\xi}\,,$$

$$\eta'' + 2\omega(\xi'\sin\phi_1 + \zeta'\cos\phi_1) - \omega^2[\eta + v_1\cos\epsilon_1 - u_1\sin\epsilon_1] = \frac{\partial V}{\partial\eta}\,,$$

$$\zeta'' - 2\omega\cos\phi_1\cdot\eta' - \omega^2\cos\phi_1\,[\xi\sin\phi_1 + \zeta\cos\phi_1 + u_1\cos\epsilon_1 + v_1\sin\epsilon_1] = \frac{\partial V}{\partial\zeta}\,,$$

If further we represent by τ the perpendicular distance of a general point P from the axis Oz, and by λ the angle which the plane POz makes with the plane uOz (see Fig. 2), the equations of transformation from the rectangular coördinates u, v, z to the cylindrical coördinates τ, λ, z are,

(12) $$u = \tau\cos\lambda\,, \quad v = \tau\sin\lambda\,, \quad z = z\,.$$

By this transformation the equations (4) are transformed into the equations:

(13)
$$\frac{d^2\tau}{dt^2} - \tau\left(\omega + \frac{d\lambda}{dt}\right)^2 = \frac{\partial V}{\partial\tau}\,, \qquad \frac{d}{dt}\left\{\tau^2\left(\omega + \frac{d\lambda}{dt}\right)\right\} = \frac{\partial V}{\partial\lambda}\,,$$
$$\frac{d^2z}{dt^2} = \frac{\partial V}{\partial z}\,.$$

Finally, we will denote by r the distance from O of the general point P and by ψ the angle which the radius vector OP makes with the plane uOv (see Fig. 2). Then the equations of transformation from the cylindrical coördinates τ, λ, z to the polar coördinates r, ψ, λ are

$$\tau = r\cos\psi\,, \quad z = r\sin\psi\,, \quad \lambda = \lambda\,. \tag{14}$$

By this transformation equations (13) are transformed into the equations:

$$\begin{aligned}
&\frac{d^2r}{dt^2} - r\left(\frac{d\psi}{dt}\right)^2 - r\left(\omega + \frac{d\lambda}{dt}\right)^2\cos^2\psi &&= \frac{\partial V}{\partial r}\,,\\
&\frac{d}{dt}\left(r^2\frac{d\psi}{dt}\right) + r^2\left(\omega + \frac{d\lambda}{dt}\right)^2\cos\psi\sin\psi &&= \frac{\partial V}{\partial\psi}\,,\\
&\frac{d}{dt}\left\{r^2\left(\omega + \frac{d\lambda}{dt}\right)\cos^2\psi\right\} &&= \frac{\partial V}{\partial\lambda}\,.
\end{aligned} \tag{15}$$

FIGURE SHOWING
DR. WOODWARD'S COÖRDINATES

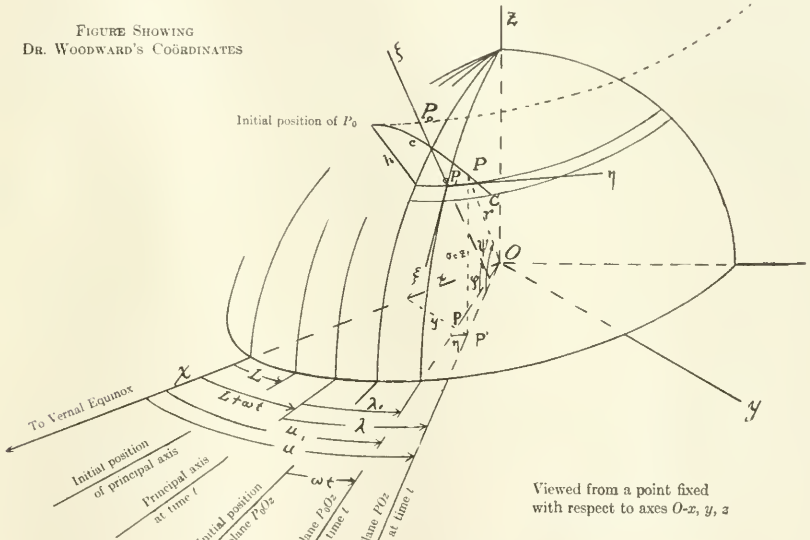

FIG. 3

where accents denote time-derivatives. Since by relation (5)

(10)
$$\begin{aligned}
\frac{\partial W}{\partial\xi} &= \frac{\partial V}{\partial\xi} + \omega^2\sin\phi_1\left[\xi\sin\phi_1 + \zeta\cos\phi_1 + u_1\cos\epsilon_1 + v_1\sin\epsilon_1\right]\,,\\
\frac{\partial W}{\partial\eta} &= \frac{\partial V}{\partial\eta} + \omega^2\left[\eta + v_1\cos\epsilon_1 - u_1\sin\epsilon_1\right]\,,\\
\frac{\partial W}{\partial\zeta} &= \frac{\partial V}{\partial\zeta} + \omega^2\cos\phi_1\left[\xi\sin\phi_1 + \zeta\cos\phi_1 + u_1\cos\epsilon_1 + v_1\sin\epsilon_1\right]\,,
\end{aligned}$$

it follows that these differential equations are the same as equations (8).

If in equations (9) we put $v_1 = 0$ and $\epsilon_1 = 0$, we obtain the equations

$$\begin{aligned}
&\xi'' - 2\omega\sin\phi_1\cdot\eta' - \omega^2\sin\phi_1\cdot u &&= \frac{\partial V}{\partial\xi}\\
&\eta'' + 2\omega\left[\xi'\sin\phi_1 + \zeta'\cos\phi_1\right] - \omega^2\eta &&= \frac{\partial V}{\partial\eta}\,,\\
&\zeta'' - 2\omega\cos\phi_1\cdot\eta' - \omega^2\cos\phi\cdot u &&= \frac{\partial V}{\partial\zeta}\,.
\end{aligned} \tag{11}$$

It is no loss in generality to assume $v_1 = 0$, but it is a loss to assume both $v_1 = 0$ and $\epsilon_1 = 0$, for by so doing we assume the astronomical meridian plane of P_1 to pass through Oz. Hence equations (11) are less general than equations (8) or (9).

Before commenting on the various equations of motion just found we will make the following definitions:

DEFINITIONS. If the distribution of the earth's gravitating matter is of such a nature that the potential functions V or W are of the form:

$$f(r, z) \quad , \quad r = \sqrt{u^2 + v^2} = \sqrt{x^2 + y^2} \quad .$$

we will say that there exists *a distribution of revolution*. When this is not the case we will call the distribution *asymmetric*.

Each of the five systems of equations (2), (4) or (6), (8), (13), (15) holds for an asymmetric distribution. For a distribution of revolution the function V satisfies identically the relation.

$$(16) \qquad \frac{\partial V}{\partial \lambda} = O.$$

Dr. WOODWARD virtually assumes a distribution of revolution when he asserts below his equations (3) (*A.J.*, Nos. 651-652, p. 19) that "λ_1 and μ_1 refer to P_0 as well as to P_1." His various systems of coördinates are represented in Fig. 3. Notwithstanding his assumption, his equations (24) and (25_1) (*A. J.*, p. 21), which are the same, respectively, as equations (15) and (4) above, hold for an asymmetric distribution. However, his equations (25) (*A.J.*, p. 21), which are the same as equations (11) above, do *not* hold for an asymmetric distribution, but hold for a distribution of revolution. (See statement below equations (11) in the foot-note to equations (8)).

Equations (8) above, which are more general than Dr. WOODWARD's equations (25) (*A.J.*, p. 21), were derived by me in my second paper (*T.A.M.S.*, Vol. XIII, pp. 469-490) by two different methods. The first of these is the same as that outlined above (see §3, p. 477) and the second made use of the equations of relative motion (See §4, p. 481). The special form which the system of equations (13) assumes for a distribution of revolution, was derived by me in my first paper (*T.A.M.S.*, Vol. XII, pp. 335-353) in §2.

Professor F. R. MOULTON in the paper referred to in the addendum to the introduction of this article, uses as his equations of motion (1) the equations (15) above. However, in the integration of these equations he assumes a distribution of revolution. He states this assumption by saying "V is a sum of zonal harmonics and consequently independent of λ." Professor MOULTON's coordinates are shown in Fig. 4 of this article.†

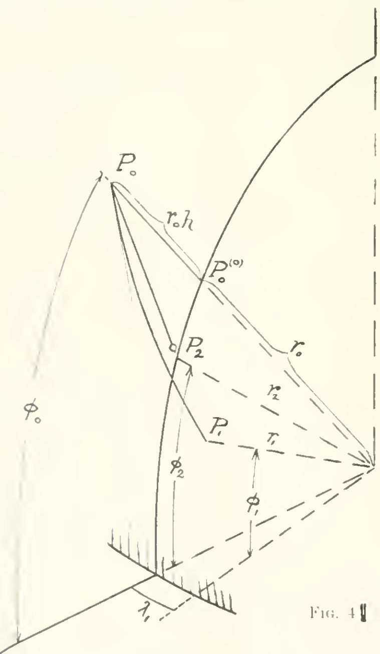

FIG. 4

†All of the equations (2), (4), (8), (13), (15) may be gotten, as Dr. WOODWARD gets his equations (24), (25), (25_1), (*A.J.*, p. 21), by the Lagrangian equation of motion:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \rho'}\right) - \frac{\partial T}{\partial \rho} = \frac{\partial V}{\partial \rho} .$$

Denoting time derivatives by accents, the kinetic energy of a unit particle in the various systems of co-ordinates, is

$$\begin{aligned} T &= \tfrac{1}{2}\,(x'^2 + y'^2 + z'^2) \\ &= \tfrac{1}{2}\left[u'^2 + v'^2 + z'^2 + 2\omega\,(uv' - vu') + \omega^2\,(u^2 + v^2)\right] \\ &= \tfrac{1}{2}\big\{\xi'^2 + \zeta'^2 + \eta'^2 + 2\omega[(\xi' \sin\phi_1 + \zeta' \cos\phi_1) \\ &\qquad (u_1 \sin\epsilon_1 - v_1 \cos\epsilon_1) + (\xi\eta' - \xi'\eta)\sin\phi_1 + (\zeta\eta' - \zeta'\eta)\cos\phi_1 \\ &\qquad + \eta'(u_1 \cos\epsilon_1 + v_1 \sin\epsilon_1)] + \omega^2[\xi^2 \sin^2\phi_1 + 2\xi\zeta \sin\phi_1 \cos\phi_1 \\ &\qquad + \zeta^2 \cos^2\phi_1 + \eta^2 + 2(\xi \sin\phi_1 + \zeta \cos\phi_1)(u_1 \cos\epsilon_1 + v_1 \sin\epsilon_1) \\ &\qquad - 2\eta\,(u_1 \sin\epsilon_1 - v_1 \cos\epsilon_1) + u_1^2 + v_1^2]\big\} \\ &= \tfrac{1}{2}\left[r'^2 + r^2\psi'^2 + r^2(\omega + \lambda')^2 \cos^2\psi\right] \\ &= \tfrac{1}{2}\left[r'^2 + z'^2 + r^2(\omega + \lambda')^2\right] \end{aligned}$$

Of these five expressions for T the last one is not given by Dr. WOODWARD, the next to last is the same as Dr. WOODWARD's expression (20), (*A.J.*, p. 21), and the second is the same as the last of Dr. WOODWARD's expressions (21) since his co-ordinates ρ, η, σ are practically the same as my co-ordinates u, v, z. However, the expression for T in terms of ξ, η, ζ which is here given is more general than the first of Dr. WOODWARD's expressions (21) because my co-ordinates ξ, η, ζ are more general than his. It leads to the equations (8) or (9), which (as already pointed out below equations (11) and (16)) are more general than Dr. WOODWARD's equations (25) (*A.J.*, p. 21.)

§3. Integration of the Equations of Motion.

Let us integrate the system of equations (8). Since the falling particle starts from a position of rest (with respect to the axes $P_1 - \eta, \xi, \zeta$) at P_0, whose distance above P_1 is h, the initial conditions are

$$(17)\quad \text{when}\quad t = 0\ ,\quad \eta = \xi = 0,\quad \zeta = h\ ,$$

$$\frac{d\eta}{dt} = \frac{d\xi}{dt} = \frac{d\zeta}{dt} = 0\ .$$

In the solution the symbols W_η^0, W_ξ^0, W_ζ^0, $W_{\eta\eta}^0$, $W_{\eta\xi}^0$, etc., will appear. These stand respectively for the values which the derivatives $\frac{\partial W}{\partial\eta}$, $\frac{\partial W}{\partial\xi}$, $\frac{\partial W}{\partial\zeta}$, $\frac{\partial^2 W}{\partial\eta^2}$, $\frac{\partial^2 W}{\partial\eta\partial\xi}$, etc., have at the point P_0.

The solution of the system of equations (8) which is subject to the conditions (17), is*

$$\left.\begin{aligned}
\eta = {} & \tfrac{1}{2}W_\eta^0 \,.\, t^2 \\
& - \tfrac{1}{3}\omega\left[\cos\phi_1 \cdot W_\zeta^0 + \sin\phi_1 \cdot W_\xi^0\right]\cdot t^3 \\
& + \tfrac{1}{24}\left[W_{\eta\eta}^0\cdot W_\eta^0 + W_{\eta\xi}^0\cdot W_\xi^0 + W_{\eta\zeta}^0\cdot W_\zeta^0 - 4\omega^2 W_\eta^0\right]\cdot t^4 \\
& - \tfrac{1}{60}\omega\big\{W_\zeta^0\cdot\left[\cos\phi_1\cdot(W_{\eta\eta}^0 + W_{\zeta\zeta}^0 - 4\omega^2) + \sin\phi_1\cdot W_{\xi\zeta}^0\right] \\
& + W_\xi^0\cdot\left[\cos\phi_1\cdot W_{\xi\zeta}^0 + \sin\phi_1\cdot(W_{\eta\eta}^0 + W_{\xi\xi}^0 - 4\omega^2)\right]\big\}\cdot t^5 \\
& + \ldots\ , \\
\xi = {} & \tfrac{1}{2}W_\xi^0\cdot t^2 \\
& + \tfrac{1}{3}\omega\sin\phi_1\cdot W_\eta^0\cdot t^3 \\
& + \tfrac{1}{24}\big[W_{\xi\eta}^0\cdot W_\eta^0 + W_{\xi\xi}^0\cdot W_\xi^0 + W_{\xi\zeta}^0\cdot W_\zeta^0 \\
& \qquad - 4\omega^2\sin\phi_1\cdot(\cos\phi_1\cdot W_\zeta^0 + \sin\phi_1\cdot W_\xi^0)\big]\cdot t^4 \\
& + \tfrac{1}{60}\omega\big\{W_\zeta^0\cdot\left[\sin\phi_1\cdot W_{\eta\zeta}^0 - \cos\phi_1\cdot W_{\xi\eta}^0\right] \\
& + W_\eta^0\cdot\left[\cos\phi_1\cdot W_{\xi\zeta}^0 + \sin\phi_1\cdot(W_{\eta\eta}^0 + W_{\xi\xi}^0 - 4\omega^2)\right]\big\}\cdot t^5 \\
& + \ldots\ , \\
\zeta = {} & h + \tfrac{1}{2}W_\zeta^0\cdot t^2 \\
& + \tfrac{1}{3}\omega\cos\phi_1\cdot W_\eta^0\cdot t^3 \\
& + \tfrac{1}{24}\big[W_{\zeta\eta}^0\cdot W_\eta^0 + W_{\zeta\xi}^0\cdot W_\xi^0 + W_{\zeta\zeta}^0\cdot W_\zeta^0 \\
& \qquad - 4\omega^2\cos\phi_1\cdot(\cos\phi_1\cdot W_\zeta^0 + \sin\phi_1\,.\,W_\xi^0)\big]\cdot t^4 \\
& + \tfrac{1}{60}\omega\big\{W_\xi^0\cdot\left[\cos\phi_1\cdot W_{\eta\xi}^0 - \sin\phi_1\cdot W_{\zeta\eta}^0\right] \\
& + W_\eta^0\left[\sin\phi_1\cdot W_{\xi\zeta}^0 + \cos\phi_1\cdot(W_{\eta\eta}^0 + W_{\zeta\zeta}^0 - 4\omega^2)\right]\big\}\cdot t^5 \\
& + \ldots\ .
\end{aligned}\right\}\quad (18)$$

---

* In order to obtain this solution we will write equations (8) in the following form:

$$(\alpha)\qquad\begin{aligned}
&\eta'' + 2\omega\,(\sin\phi_1\cdot\xi' + \cos\phi_1\cdot\zeta') - W_\eta = 0\ ,\\
&\xi'' - 2\omega\sin\phi_1\cdot\eta' \qquad\qquad\quad - W_\xi = 0\ ,\\
&\zeta'' - 2\omega\cos\phi_1\cdot\eta' \qquad\qquad\quad - W_\zeta = 0\ ,
\end{aligned}$$

where the primes ($'$) and seconds ($''$) denote the first and second derivatives, with respect to the time t, of the functions to which they are attached, and the subscripts (η, ξ, ζ) denote the first partial derivatives of W with respect to the attached subscript. In the equations which follow thirds ($'''$), etc., and double subscripts $(\eta\eta$, $\eta\xi \cdots)$, etc., will be used to denote the higher derivatives. For the particular values of η, ξ, ζ, and their derivatives with respect to t, which correspond to $t = 0$, we shall use the symbols, η_0, ξ_0, ζ_0, η_0', ξ_0', ζ_0', η_0'', ξ_0'', etc. Hence condition (17) may be written in the following form:

$$(\beta)\qquad \eta_0 = \xi_0 = 0\ ,\qquad \zeta_0 = h\ ,\qquad \eta_0' = \xi_0' = \zeta_0' = 0\ .$$

For the particular values of the derivatives of W which correspond to $\eta = \eta_0$, $\xi = \xi_0$, $\zeta = \zeta_0$ we shall use the symbols W_η^0, W_ξ^0, W_ζ^0, $W_{\eta\eta}^0$, $W_{\eta\xi}^0$, $W_{\eta\zeta}^0$, $W_{\xi\xi}^0$, $W_{\xi\zeta}^0$, etc.

For a set of functions

$$\eta = \lambda\,(t)\ ,\qquad \xi = \mu\,(t)\ ,\qquad \zeta = \nu\,(t)\ ,$$

which is a solution of the set of equations (α), the left hand member of each of the equations (α), vanishes identically when regarded as a function of the time t. Therefore, for this solution, the derivative with respect to t of each of the left hand members of equations (α) must also vanish identically. Thus we obtain the identities:

(α')

$$\begin{aligned}
&\eta''' + 2\omega\,(\sin\phi_1\cdot\xi'' + \cos\phi_1\cdot\zeta'') \\
&\qquad\qquad - (W_{\eta\eta}\cdot\eta' + W_{\eta\xi}\cdot\xi' + W_{\eta\zeta}\cdot\zeta') = 0\ ,\\
&\xi''' - 2\omega\sin\phi_1\cdot\eta'' - (W_{\xi\eta}\cdot\eta' + W_{\xi\xi}\cdot\xi' + W_{\xi\zeta}\cdot\zeta') = 0\ ,\\
&\zeta''' - 2\omega\cos\phi_1\cdot\eta'' - (W_{\zeta\eta}\cdot\eta' + W_{\zeta\xi}\cdot\xi' + W_{\zeta\zeta}\cdot\zeta') = 0\ .
\end{aligned}$$

Similarly, if we assume the existence and continuity of all the derivatives which are needed, we obtain by successive differentiations with respect to t the following identities:

(α'')

$$\begin{aligned}
&\eta^{iv} + 2\omega(\sin\phi_1\cdot\xi''' + \cos\phi_1\cdot\zeta''') - (W_{\eta\eta}\cdot\eta'' + W_{\eta\xi}\cdot\xi'' + W_{\eta\zeta}\cdot\zeta'') \\
&\qquad - \left(\eta'\frac{d}{dt}W_{\eta\eta} + \xi'\frac{d}{dt}W_{\eta\xi} + \zeta'\frac{d}{dt}W_{\eta\zeta}\right) = 0\ ,\\
&\xi^{iv} - 2\omega\sin\phi_1\cdot\eta''' - (W_{\xi\eta}\cdot\eta'' + W_{\xi\xi}\cdot\xi'' + W_{\xi\zeta}\cdot\zeta'') \\
&\qquad - \left(\eta'\frac{d}{dt}W_{\xi\eta} + \xi'\frac{d}{dt}W_{\xi\xi} + \zeta'\frac{d}{dt}W_{\xi\zeta}\right) = 0\ ,\\
&\zeta^{iv} - 2\omega\cos\phi_1\cdot\eta''' - (W_{\zeta\eta}\cdot\eta'' + W_{\zeta\xi}\cdot\xi'' + W_{\zeta\zeta}\cdot\zeta'') \\
&\qquad - \left(\eta'\frac{d}{dt}W_{\zeta\eta} + \xi'\frac{d}{dt}W_{\zeta\xi} + \zeta'\frac{d}{dt}W_{\zeta\zeta}\right)\ ;
\end{aligned}$$

(α''')

$$\begin{aligned}
&\eta^{v} + 2\omega(\sin\phi_1\cdot\xi^{iv} + \cos\phi_1\cdot\zeta^{iv}) - (W_{\eta\eta}\cdot\eta''' + W_{\eta\xi}\cdot\xi''' + W_{\eta\zeta}\cdot\zeta''') \\
&\qquad - 2\left(\eta''\frac{d}{dt}W_{\eta\eta} + \xi''\frac{d}{dt}W_{\eta\xi} + \zeta''\frac{d}{dt}W_{\eta\zeta}\right) \\
&\qquad - \left(\eta'\frac{d^2}{dt^2}W_{\eta\eta} + \xi'\frac{d^2}{dt^2}W_{\eta\xi} + \zeta'\frac{d^2}{dt^2}W_{\eta\zeta}\right) = 0\ ,\\
&\xi^{v} - 2\omega\sin\phi_1\cdot\eta^{iv} - (W_{\xi\eta}\cdot\eta''' + W_{\xi\xi}\cdot\xi''' + W_{\xi\zeta}\cdot\zeta''') \\
&\qquad - 2\left(\eta''\frac{d}{dt}W_{\xi\eta} + \xi''\frac{d}{dt}W_{\xi\xi} + \zeta''\frac{d}{dt}W_{\xi\zeta}\right) \\
&\qquad - \left(\eta'\frac{d^2}{dt^2}W_{\xi\eta} + \xi'\frac{d^2}{dt^2}W_{\xi\xi} + \zeta'\frac{d^2}{dt^2}W_{\xi\zeta}\right) = 0\ .\\
&\zeta^{v} - 2\omega\cos\phi_1\cdot\eta^{iv} - (W_{\zeta\eta}\cdot\eta''' + W_{\zeta\xi}\cdot\xi''' + W_{\zeta\zeta}\cdot\zeta''') \\
&\qquad - 2\left(\eta''\frac{d}{dt}W_{\zeta\eta} + \xi''\frac{d}{dt}W_{\zeta\xi} + \xi''\frac{d}{dt}\cdot W_{\zeta\zeta}\right) \\
&\qquad - \left(\eta'\frac{d^2}{dt^2}W_{\zeta\eta} + \xi''\frac{d^2}{dt^2}W_{\zeta\xi} + \zeta''\frac{d^2}{dt^2}W_{\zeta\zeta}\right) = 0\ .
\end{aligned}$$

These are the parametric equations of the curve Γ (Fig. 1). From them it is evident that the tangent to Γ at P_0 is the vertical P_0T of P_0.

From Definitions 1 and 2 (Part II, § 1) it follows that the easterly and southerly (meridional) deviations of the falling particle are the values of η and ξ, respectively, for the particular value of t for which $\zeta = 0$. Let us denote this value by $\bar{t}$ and the corresponding values of η and ξ by $\bar{\eta}$ and $\bar{\xi}$ respectively.

From these identities we can find the values of the constants η_0, ξ_0, ζ_0, η_0', ξ_0', ζ_0', η_0'', ξ_0'', ζ_0'', etc. From conditions (β) we already know that

$$\eta_0 = 0, \quad \xi_0 = 0, \quad \zeta_0 = h, \qquad \eta_0' = 0, \quad \xi_0' = 0, \quad \zeta_0' = 0 .$$

Therefore when $t = 0$ the identities (α) yield the relations

$$\eta_0'' = W_\eta^0 , \quad \xi_0'' = W_\xi^0 , \quad \zeta_0'' = W_\zeta^0 ;$$

the identities (α') the relations

$$\begin{aligned} \eta_0''' &= -2\omega\,(\sin\phi_1 \cdot W_\xi^0 + \cos\phi_1 \cdot W_\zeta^0) , \\ \xi_0''' &= +2\omega\,\sin\phi_1 \cdot W_\eta^0 , \\ \zeta_0''' &= +2\omega\,\cos\phi_1 \cdot W_\eta^0 ; \end{aligned}$$

the identities (α'') the relations

$$\begin{aligned} \eta_0^{iv} &= -2\omega(\sin\phi_1 \cdot 2\omega\sin\phi_1 \cdot W_\eta^0 + \cos\phi_1 \cdot 2\omega\cos\phi_1 \cdot W_\eta^0) \\ &\qquad + W_{\eta\eta}^0 \cdot W_\eta^0 + W_{\eta\xi}^0 \cdot W_\xi^0 + W_{\eta\zeta}^0 \cdot W_\zeta^0 \\ &= -4\omega^2 \cdot W_\eta^0 + W_{\eta\eta}^0 \cdot W_\eta^0 + W_{\eta\xi}^0 \cdot W_\xi^0 + W_{\eta\zeta}^0 \cdot W_\zeta^0 , \end{aligned}$$

$$\begin{aligned} \xi_0^{iv} &= +2\omega\sin\phi_1 \cdot (-2\omega)\,(\sin\phi_1 \cdot W_\xi^0 + \cos\phi_1 \cdot W_\zeta^0) \\ &\qquad + W_{\xi\eta}^0 \cdot W_\eta^0 + W_{\xi\xi}^0 \cdot W_\xi^0 + W_{\xi\zeta}^0 \cdot W_\zeta^0 \\ &= -4\omega^2\sin\phi_1 \cdot (\sin\phi_1 \cdot W_\xi^0 + \cos\phi_1 \cdot W_\zeta^0) \\ &\qquad + W_{\xi\eta}^0 \cdot W_\eta^0 + W_{\xi\xi}^0 \cdot W_\xi^0 + W_{\xi\zeta}^0 \cdot W_\zeta^0 , \end{aligned}$$

$$\begin{aligned} \zeta_0^{iv} &= +2\omega\cos\phi_1 \cdot (-2\omega)\,(\sin\phi_1 \cdot W_\xi^0 + \cos\phi_1 \cdot W_\zeta^0) \\ &\qquad + W_{\zeta\eta}^0 \cdot W_\eta^0 + W_{\zeta\xi}^0 \cdot W_\xi^0 + W_{\zeta\zeta}^0 \cdot W_\zeta^0 \\ &= -4\omega^2\cos\phi_1 \cdot (\sin\phi_1 \cdot W_\xi^0 + \cos\phi_1 \cdot W_\zeta^0) \\ &\qquad + W_{\zeta\eta}^0 \cdot W_\eta^0 + W_{\zeta\xi}^0 \cdot W_\xi^0 + W_{\zeta\zeta}^0 \cdot W_\zeta^0 ; \end{aligned}$$

and the identities (α''') the relations

$$\begin{aligned} \eta_0^{v} &= -2\omega\,[\sin\phi_1 \cdot \{-4\omega^2\sin\phi_1 \cdot (\sin\phi_1 \cdot W_\xi^0 + \cos\phi_1\, W_\zeta^0) \\ &\qquad + W_{\xi\eta}^0 \cdot W_\eta^0 + W_{\xi\xi}^0 \cdot W_\xi^0 + W_{\xi\zeta}^0 \cdot W_\zeta^0\} \\ &\qquad + \cos\phi_1 \cdot \{-4\omega^2\cos\phi_1 \cdot (\sin\phi_1 \cdot W_\xi^0 + \cos\phi_1\, W_\zeta^0) \\ &\qquad + W_{\zeta\eta}^0 \cdot W_\eta^0 + W_{\zeta\xi}^0 \cdot W_\xi^0 + W_{\zeta\zeta}^0 \cdot W_\zeta^0\}] \\ &\qquad + W_{\eta\eta}^0 \cdot (-2\omega)\,(\sin\phi_1 \cdot W_\xi^0 + \cos\phi_1 \cdot W_\zeta^0) \\ &\qquad + W_{\eta\xi}^0 \cdot 2\omega\sin\phi_1 \cdot W_\eta^0 + W_{\eta\zeta}^0 \cdot 2\omega\cos\phi_1 \cdot W_\eta^0 \\ &= 8\omega^3\,(\sin\phi_1 \cdot W_\xi^0 + \cos\phi_1 \cdot W_\zeta^0) \\ &\qquad - 2\omega\,[W_\xi^0\,(\sin\phi_1 \cdot W_{\xi\xi}^0 + \sin\phi_1 \cdot W_{\eta\eta}^0 + \cos\phi_1 \cdot W_{\zeta\xi}^0) \\ &\qquad + W_\zeta^0\,(\cos\phi_1 \cdot W_{\zeta\zeta}^0 + \cos\phi_1 \cdot W_{\eta\eta}^0 + \sin\phi_1 \cdot W_{\xi\zeta}^0)] , \end{aligned}$$

$$\begin{aligned} \xi_0^{v} &= +2\omega\sin\phi_1 \cdot [-4\omega^2 W_\eta^0 + W_{\eta\eta}^0 \cdot W_\eta^0 + W_{\eta\xi}^0 \cdot W_\xi^0 + W_{\eta\zeta}^0 \cdot W_\zeta^0] \\ &\qquad + W_{\xi\eta}^0\,(-2\omega)\,(\sin\phi_1 \cdot W_\xi^0 + \cos\phi_1 \cdot W_\zeta^0) \\ &\qquad + W_{\xi\xi}^0 \cdot 2\omega\sin\phi_1 \cdot W_\eta^0 + W_{\xi\zeta}^0 \cdot 2\omega\cos\phi_1 \cdot W_\eta^0 \\ &= -8\omega^3\sin\phi_1 \cdot W_\eta^0 \\ &\qquad + 2\omega\,[W_\eta^0\,(\sin\phi_1 \cdot W_{\eta\eta}^0 + \sin\phi_1 \cdot W_{\xi\xi}^0 + \cos\phi_1 \cdot W_{\xi\zeta}^0) \\ &\qquad\qquad + W_\zeta^0\,(\sin\phi_1 \cdot W_{\eta\zeta}^0 - \cos\phi_1 \cdot W_{\xi\eta}^0)] , \end{aligned}$$

$$\begin{aligned} \zeta_0^{v} &= +2\omega\cos\phi_1 \cdot [-4\omega^2 \cdot W_\eta^0 + W_{\eta\eta}^0 \cdot W_\eta^0 + W_{\eta\xi}^0 \cdot W_\xi^0 \\ &\qquad + W_{\eta\zeta}^0 \cdot W_\zeta^0] + W_{\zeta\eta}^0\,(-2\omega)\,(\sin\phi_1 \cdot W_\xi^0 + \cos\phi_1 \cdot W_\zeta^0) \\ &\qquad + W_{\zeta\xi}^0 \cdot 2\omega\sin\phi_1 \cdot W_\eta^0 + W_{\zeta\zeta}^0 \cdot 2\omega\cos\phi_1 \cdot W_\eta^0 \\ &= -8\omega^3\cos\phi_1 \cdot W_\eta^0 \\ &\qquad + 2\omega\,[W_\eta^0\,(\cos\phi_1 \cdot W_{\eta\eta}^0 + \cos\phi_1 \cdot W_{\zeta\zeta}^0 + \sin\phi_1 \cdot W_{\xi\zeta}^0) \\ &\qquad + W_\xi^0\,(\cos\phi_1 \cdot W_{\eta\xi}^0 - \sin\phi_1 \cdot W_{\eta\zeta}^0)] . \end{aligned}$$

It should be noted that for $t = 0$ the expressions

$$\frac{d}{dt}W_{\eta\eta} , \quad \frac{d}{dt}W_{\eta\xi} , \quad \frac{d}{dt}W_{\eta\zeta} , \quad \frac{d}{dt}W_{\xi\xi} , \quad \frac{d}{dt}W_{\xi\zeta} , \quad \frac{d}{dt}W_{\zeta\zeta}$$

which occur in the identities (α''') all vanish. For

$$\frac{d}{dt}W_{\eta\eta} = W_{\eta\eta\eta} \cdot \eta' + W_{\eta\eta\xi} \cdot \xi' + W_{\eta\eta\zeta} \cdot \zeta' , \quad \text{etc.,}$$

and for $t = 0$, $\eta' = \xi' = \zeta' = 0$.

Let us now assume that the conditions are satisfied under which the set of differential equations (8) or (α) have a solution of the form

$$\begin{aligned} \eta &= a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + \cdots , \\ \xi &= b_0 + b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4 + b_5 t^5 + \cdots , \\ \zeta &= c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5 + \cdots , \end{aligned} \qquad (\gamma)$$

in the neighborhood of the point P_0, ($\xi = 0$, $\eta = 0$, $\zeta = h$). It then follows from the preceding work that for the initial conditions (17) or (β),

$$\begin{aligned} &a_0 = \eta_0 = 0 , \qquad b_0 = \xi_0 = 0 , \qquad c_0 = \zeta_0 = h , \\ &a_1 = \eta_0' = 0 , \qquad b_1 = \xi_0' = 0 , \qquad c_1 = \zeta_0' = 0 , \\ &a_2 = \tfrac{1}{2}\eta_0'' = \tfrac{1}{2}W_\eta^0 , \quad b_2 = \tfrac{1}{2}\xi_0'' = \tfrac{1}{2}W_\xi^0 , \quad c_2 = \tfrac{1}{2}\zeta_0'' = \tfrac{1}{2}W_\zeta^0 , \end{aligned}$$

$$\begin{aligned} a_3 &= \tfrac{1}{6}\eta_0''' = -\tfrac{1}{3}\omega\,[\sin\phi_1 \cdot W_\xi^0 + \cos\phi_1 \cdot W_\zeta^0] , \\ b_3 &= \tfrac{1}{6}\xi_0''' = +\tfrac{1}{3}\omega\sin\phi_1 \cdot W_\eta^0 , \\ c_3 &= \tfrac{1}{6}\zeta_0''' = +\tfrac{1}{3}\omega\cos\phi_1 \cdot W_\eta^0 , \end{aligned}$$

$$\begin{aligned} a_4 &= \tfrac{1}{24}\eta_0^{iv} = \tfrac{1}{24}[W_{\eta\eta}^0 \cdot W_\eta^0 + W_{\eta\xi}^0 \cdot W_\xi^0 + W_{\eta\zeta}^0 \cdot W_\zeta^0 - 4\omega^2 \cdot W_\eta^0] , \\ b_4 &= \tfrac{1}{24}\xi_0^{iv} = \tfrac{1}{24}[W_{\xi\eta}^0 \cdot W_\eta^0 + W_{\xi\xi}^0 \cdot W_\xi^0 + W_{\xi\zeta}^0 \cdot W_\zeta^0 \\ &\qquad - 4\omega^2\sin\phi_1 \cdot (\sin\phi_1 \cdot W_\xi^0 + \cos\phi_1 \cdot W_\zeta^0)] , \\ c_4 &= \tfrac{1}{24}\zeta_0^{iv} = \tfrac{1}{24}[W_{\zeta\eta}^0 \cdot W_\eta^0 + W_{\zeta\xi}^0 \cdot W_\xi^0 + W_{\zeta\zeta}^0 \cdot W_\zeta^0 \\ &\qquad - 4\omega^2\cos\phi_1 \cdot (\sin\phi_1 \cdot W_\xi^0 + \cos\phi_1 \cdot W_\zeta^0)] , \end{aligned}$$

$$\begin{aligned} a_5 &= \tfrac{1}{120}\eta_0^{v} = -\tfrac{1}{60}\omega\{W_\xi^0[\cos\phi_1 \cdot W_{\xi\zeta}^0 + \sin\phi_1 \cdot (W_{\eta\eta}^0 + W_{\xi\xi}^0 - 4\omega^2)] \\ &\qquad + W_\zeta^0[\sin\phi_1 \cdot W_{\xi\zeta}^0 + \cos\phi_1 \cdot (W_{\eta\eta}^0 + W_{\zeta\zeta}^0 - 4\omega^2)]\} , \\ b_5 &= \tfrac{1}{120}\xi_0^{v} = +\tfrac{1}{60}\omega\{W_\eta^0[\cos\phi_1 \cdot W_{\xi\zeta}^0 + \sin\phi_1 \cdot (W_{\eta\eta}^0 + W_{\xi\xi}^0 - 4\omega^2)] \\ &\qquad + W_\zeta^0[\sin\phi_1 \cdot W_{\eta\zeta}^0 - \cos\phi_1 \cdot W_{\xi\eta}^0]\} , \\ c_5 &= \tfrac{1}{120}\zeta_0^{v} = +\tfrac{1}{60}\omega\{W_\eta^0[\sin\phi_1 \cdot W_{\xi\zeta}^0 + \cos\phi_1 \cdot (W_{\eta\eta}^0 + W_{\zeta\zeta}^0 - 4\omega^2)] \\ &\qquad + W_\xi^0[\cos\phi_1 \cdot W_{\eta\xi}^0 - \sin\phi_1 \cdot W_{\eta\zeta}^0]\} . \end{aligned}$$

Thus are obtained the coefficients of the solution (18).

This method of getting the integrals (18) is the same as that given in § 5 of my second paper (*T.A.M.S.*, Vol. XIII, pp. 469–490), except that here W_η^0 and W_ξ^0 are not zero.

In order to express the easterly and southerly deviations $\bar{\eta}$ and $\bar{\xi}$ in terms of the time of fall $\bar{t}$, we must examine carefully the coefficients of the series (18). For this purpose we will make use of the fact that *the quantities* W_η, W_ξ, W_ζ, *layed off respectively on the axes* $P_1 - \eta, \xi, \zeta$, *are the components of the vector which represents the acceleration* g *at the general point* P, *and this vector lies along the vertical of* P. Accordingly if we denote by $W_\eta^{(1)}$, $W_\xi^{(1)}$, $W_\zeta^{(1)}$ and g_1 the values of W_η, W_ξ, W_ζ and g, respectively, at P_1, we have, since $P_1 - \eta, \xi, \zeta$ are the cardinal axes of P_1,

$$(19) \qquad W_\eta^{(1)} = W_\xi^{(1)} = 0\,, \qquad W_\zeta^{(1)} = -\,g_1\,.$$

Therefore, since the coördinates of P_1 and P_0 are (o, o, o) and (o, o, h) respectively,*

(20)

$$W_\eta^0 = W_{\eta\zeta}^{(1)} \cdot h\,, \qquad W_{\eta\eta}^0 = W_{\eta\eta}^{(1)} + W_{\eta\eta\zeta}^{(1)} \cdot h\,,$$

$$W_{\xi\zeta}^0 = W_{\xi\zeta}^{(1)} + W_{\xi\zeta\zeta}^{(1)} \cdot h\,,$$

$$W_\xi^0 = W_{\xi\zeta}^{(1)} \cdot h\,, \qquad W_{\xi\xi}^0 = W_{\xi\xi}^{(1)} + W_{\xi\xi\zeta}^{(1)} \cdot h\,,$$

$$W_{\zeta\eta}^0 = W_{\zeta\eta}^{(1)} + W_{\zeta\eta\zeta}^{(1)} \cdot h\,,$$

$$W_\zeta^0 = W_\zeta^{(1)} + W_{\zeta\zeta}^{(1)} \cdot h\,, \qquad W_{\zeta\zeta} = W_{\zeta\zeta}^{(1)} + W_{\zeta\zeta\zeta}^{(1)} \cdot h\,,$$

$$W_{\eta\xi}^0 = W_{\eta\xi}^{(1)} + W_{\eta\xi\zeta}^{(1)} \cdot h\,,$$

plus higher powers in h.

Hence equations (18) may be written as follows:

$$(21) \left\{ \begin{aligned} \eta = & + \tfrac{1}{2}\, W_{\eta\zeta}^{(1)} \cdot h \cdot t^2 \\ & - \tfrac{1}{3}\,\omega \cos\phi_1 \cdot W_\zeta^{(1)} \cdot t^3 \\ & - \tfrac{1}{3}\,\omega\,[\cos\phi_1 \cdot W_{\zeta\zeta}^{(1)} + \sin\phi_1 \cdot W_{\xi\zeta}^{(1)}] \cdot h \cdot t^3 \\ & + \tfrac{1}{24}\, W_{\eta\zeta}^{(1)} \cdot W_\zeta^{(1)} \cdot t^4 \\ & - \tfrac{1}{60}\,\omega\, W_\zeta^{(1)}\,[\cos\phi_1 \cdot (W_{\eta\eta}^{(1)} + W_{\zeta\zeta}^{(1)} - 4\omega^2) \\ & + \sin\phi_1 \cdot W_{\xi\zeta}^{(1)}] \cdot t^5 + \cdots\,. \\ \xi = & + \tfrac{1}{2}\, W_{\xi\zeta}^{(1)} \cdot h \cdot t^2 \\ & + \tfrac{1}{3}\,\omega \sin\phi_1 \cdot W_{\zeta\zeta}^{(1)} \cdot h \cdot t^3 \\ & + \tfrac{1}{24}\, W_\zeta^{(1)}\,[W_{\xi\zeta}^{(1)} - 4\omega^2 \sin\phi_1 \cdot \cos\phi_1] \cdot t^4 \\ & + \tfrac{1}{60}\,\omega \cdot W_\zeta^{(1)}\,[\sin\phi_1 \cdot W_{\zeta\zeta}^{(1)} - \cos\phi_1 \cdot W_{\xi\eta}^{(1)}] \cdot t^5 \\ & + \cdots\,. \\ \zeta = & \quad h + \tfrac{1}{2}\, W_\zeta^{(1)} \cdot t^2 + \tfrac{1}{2}\, W_{\zeta\zeta}^{(1)} \cdot h \cdot t^2 \\ & + \tfrac{1}{3}\,\omega \cos\phi_1 \cdot W_{\eta\zeta}^{(1)} \cdot h \cdot t^3 \\ & + \tfrac{1}{24}\, W_\zeta^{(1)} \cdot [W_{\zeta\zeta}^{(1)} - 4\omega^2 \cos^2\phi_1] \cdot t^4 + \cdots\,. \end{aligned} \right.$$

Since $\zeta = 0$ when $t = \bar{t}$, we find from the last equation

$$h = -\frac{\tfrac{1}{2} W_\zeta^{(1)} \cdot \bar{t}^2 + \tfrac{1}{24} W_\zeta^{(1)} \cdot [W_{\zeta\zeta}^{(1)} - 4\omega^2 \cos^2\phi_1] \cdot \bar{t}^4 + O\bar{t}^5}{1 + \tfrac{1}{2} W_{\zeta\zeta}^{(1)} \cdot \bar{t}^2 + \tfrac{1}{3}\omega \cos\phi_1 \cdot W_{\eta\zeta}^{(1)} \cdot \bar{t}^3 + \cdots} \tag{22}$$

$$\begin{aligned} = & -\tfrac{1}{2} W_\zeta^{(1)} \cdot \bar{t}^2 + \tfrac{1}{24} W_\zeta^{(1)}\,[5 W_{\zeta\zeta}^{(1)} + 4\omega^2 \cos^2\phi_1] \cdot \bar{t}^4 \\ & + \tfrac{1}{6}\omega\, W_\zeta^{(1)} \cdot W_{\eta\zeta}^{(1)} \cos\phi_1 \cdot \bar{t}^5 + \cdots\,. \end{aligned}$$

Hence (23)

$$\left. \begin{aligned} \bar{\eta} = & -\tfrac{1}{3}\omega \cos\phi_1 \cdot W_\zeta^{(1)} \cdot \bar{t}^3 - \tfrac{5}{24} W_\zeta^{(1)} \cdot W_{\eta\zeta}^{(1)} \cdot \bar{t}^4 \\ & + \tfrac{1}{60}\omega\, W_\zeta^{(1)}\,[9 \sin\phi_1 \cdot W_{\xi\zeta}^{(1)} + \cos\phi_1\,(9 W_{\zeta\zeta}^{(1)} - W_{\eta\eta}^{(1)} + 4\omega^2)]\,\bar{t}^5, \\ \bar{\xi} = & -\tfrac{1}{24} W_\zeta^{(1)} \cdot [5 W_{\xi\zeta}^{(1)} + 4\omega^2 \sin\phi_1 \cos\phi_1]\,\bar{t}^4 \\ & - \tfrac{1}{60}\omega\, W_\zeta^{(1)} \cdot [9 \sin\phi_1 \cdot W_{\zeta\zeta}^{(1)} + \cos\phi_1 \cdot W_{\xi\eta}^{(1)}] \cdot \bar{t}^5\,, \end{aligned} \right\}$$

and

$$\bar{\zeta} = 0\,.$$

Equations (23) give the easterly and southerly (meridional) deviations $\bar{\eta}$ and $\bar{\xi}$ in terms of the time of fall $\bar{t}$. Equation (22) gives the height of fall h in terms of the time of fall.*

---

*If $\qquad u = f(\eta, \xi, \zeta)\,,$

$$u = u^{(1)} + u_\eta^{(1)} \cdot (\eta - \eta_1) + u_\xi^{(1)} \cdot (\xi - \xi_1) + u_\zeta^{(1)} \cdot (\zeta - \zeta_1) + \cdots\,,$$

and if, in particular, $\eta_1 = \xi_1 = \zeta_1 = 0$,

$$u = u^{(1)} + u_\eta^{(1)} \cdot \eta + u_\xi^{(1)} \cdot \xi + u_\zeta^{(1)} \cdot \zeta + \cdots\,.$$

Finally if $\eta = \eta_0 = 0$, $\xi = \xi_0 = 0$, $\zeta = \zeta_0 = h$,

$$u^0 = u^{(1)} + u_\zeta^{(1)} \cdot h + \cdots\,.$$

* The coefficients of the powers of $\bar{t}$ in equations (22) and (23) are expressed in terms of the astronomical latitude of P_1 and the values at P_1 of the derivatives of W taken along the cardinal axes of P_1. In order to express these coefficients in terms of the astronomical latitude of P_0 and the values at P_0 of the derivatives of W taken along the cardinal axes of P_0, we need, in addition to relations (20), the relation between ϕ_0 and ϕ_1, and the equations of transformation from the cardinal axes of P_0 to those of P_1. The astronomical latitude ϕ of a general point P is given by the relation

$$\sin\phi = \frac{-\,W_z}{\sqrt{W_x^2 + W_y^2 + W_z^2}} = \frac{W_\xi \cos\phi_1 - W_\zeta \sin\phi_1}{\sqrt{W_\eta^2 + W_\xi^2 + W_\zeta^2}}\,.$$

If we denote this function of η, ξ, ζ by L, we have, as in relations (20),

$$L^0 = L^{(1)} + L_\zeta^{(1)} \cdot h\,.$$

where

$$L_\zeta^{(1)} = \left[\frac{\partial}{\partial\zeta}\sin\phi\right]_{(P_1)} = -\cos\phi_1 \cdot \frac{W_{\xi\zeta}^{(1)}}{W_\zeta^{(1)}}\,, \text{ since } W_\eta^{(1)} = W_\xi^{(1)} = 0$$

Hence

$$(24_1) \qquad \frac{\sin\phi_0 - \sin\phi_1}{\cos\phi_1} = -\,\frac{W_{\xi\zeta}^{(1)}}{W_\zeta^{(1)}} \cdot h$$

But, to a sufficient degree of precision,

$$(24_2) \quad \phi_0 - \phi_1 = \frac{\sin\phi_0 - \sin\phi_1}{\cos\phi_1} = -\,\frac{\cos\phi_0 - \cos\phi_1}{\sin\phi_1}\,,$$

and therefore

$$\left. \begin{aligned} \cos\phi_1 &= \cos\phi_0 - \sin\phi_1 \cdot \frac{W_{\xi\zeta}^{(1)}}{W_\zeta^{(1)}} \cdot h\,, \\ \sin\phi_1 &= \sin\phi_0 + \cos\phi_1 \cdot \frac{W_{\xi\zeta}^{(1)}}{W_\zeta^{(1)}} \cdot h\,. \end{aligned} \right\} \tag{24}$$

Let us denote by $\dot{\eta}, \dot{\xi}, \dot{\zeta}$ the co-ordinates measured along the cardinal axes of P_0, and by $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3, \gamma_3)$ the

By eliminating t between equation (22) and each of the equations (23), we can express $\bar\eta$ and $\bar\xi$ in terms of h. This elimination may be performed as follows. Equation (22) may be written in the form

$$\frac{h}{a} = \bar t^2 + b\,\bar t^4 + c\,\bar t^5 \;.$$

where

$$a = -\tfrac{1}{2}W_{\zeta}^{(1)} \;, \qquad b = -\tfrac{1}{12}(5W_{\zeta\zeta}^{(1)} + 4\omega^2\cos^2\phi_1) \;,$$
$$c = -\tfrac{1}{3}\omega\, W_{\zeta}^{(1)}\cos\phi_1 \;.$$

Hence

$$\pm\sqrt{\frac{h}{a}} = \sqrt{\bar t^2 + b\,\bar t^4 + c\,\bar t^5 + \cdots}$$
$$= \bar t + \tfrac{1}{2}b\,\bar t^3 + \tfrac{1}{2}c\,\bar t^4 + \cdots \;.$$

direction angles of the axes $P_0\dot\eta$, $P_0\dot\xi$, $P_0\dot\zeta$ respectively, with respect to axes $P_1-\eta$, ξ, ζ. See Fig. 5.

Then

$$
\begin{aligned}
\dot\eta &= \eta\cos\alpha_1 + \xi\cos\beta_1 + (\zeta - h)\cos\gamma_1 \;,\\
(25)\qquad \dot\xi &= \eta\cos\alpha_2 + \xi\cos\beta_2 + (\zeta - h)\cos\gamma_2 \;,\\
\dot\zeta &= \eta\cos\alpha_3 + \xi\cos\beta_3 + (\zeta - h)\cos\gamma_3 \;.
\end{aligned}
$$

where

(26)

$$\cos\alpha_1 = 1 \;, \qquad \cos\beta_1 = -\tan\phi_1\,\frac{W_{\eta\zeta}^{(1)}}{W_{\zeta}^{(1)}}\,h \;,$$

$$\cos\gamma_1 = -\frac{W_{\eta\zeta}^{(1)}}{W_{\zeta}^{(1)}}\cdot h \;,$$

$$\cos\alpha_2 = \tan\phi_1\cdot\frac{W_{\eta\zeta}^{(1)}}{W_{\zeta}^{(1)}}\,h \;, \qquad \cos\beta_2 = 1 \;,$$

$$\cos\gamma_2 = -\frac{W_{\xi\zeta}^{(1)}}{W_{\zeta}^{(1)}}\cdot h \;,$$

$$\cos\alpha_3 = \frac{W_{\eta\zeta}^{(1)}}{W_{\zeta}^{(1)}}\cdot h \;, \qquad \cos\beta_3 = \frac{W_{\xi\zeta}^{(1)}}{W_{\zeta}^{(1)}}\,h \;,$$

$$\cos\gamma_3 = 1 \;.$$

For the purpose of obtaining these cosines let us recall that, at a general point P, (1) the vertical is the normal to the level surface which passes through P, (2) the east-and-west line is the intersection of the horizontal plane of P by the plane ($z =$ const.) which passes through P and is perpendicular to the axis of rotation of the earth, (3) the north-and-south line is the common perpendicular to the vertical and the east-and-west line. Hence for the point P_0

$$\cos\alpha_3 : \cos\beta_3 : \cos\gamma_3 = W_{\eta}^{0} : W_{\xi}^{0} : W_{\zeta}^{0}$$
$$= W_{\eta\zeta}^{(1)}\cdot h : W_{\xi\zeta}^{(1)}\cdot h : W_{\zeta}^{(1)} + W_{\zeta\zeta}^{(1)}\cdot h$$

(by eqs. 20),

$$\cos\alpha_1 : \cos\beta_1 : \cos\gamma_1 =$$

$$\begin{vmatrix} W_{\xi}^{0} & W_{\zeta}^{0} \\ \left(\frac{\partial z}{\partial\xi}\right)_0 & \left(\frac{\partial z}{\partial\zeta}\right)_0 \end{vmatrix} : \begin{vmatrix} W_{\zeta}^{0} & W_{\eta}^{0} \\ \left(\frac{\partial z}{\partial\zeta}\right)_0 & \left(\frac{\partial z}{\partial\eta}\right)_0 \end{vmatrix} : \begin{vmatrix} W_{\eta}^{0} & W_{\xi}^{0} \\ \left(\frac{\partial z}{\partial\eta}\right)_0 & \left(\frac{\partial z}{\partial\xi}\right)_0 \end{vmatrix}$$

$$= W_{\xi}^{0}\sin\phi_1 + W_{\zeta}^{0}\cos\phi_1 : -W_{\eta}^{0}\sin\phi_1 : -W_{\eta}^{0}\cos\phi_1$$

(by eqs. 7),

$$= W_{\zeta}^{(1)}\cos\phi_1 + (W_{\zeta\zeta}^{(1)}\cos\phi_1 + W_{\xi\zeta}^{(1)}\sin\phi_1)\,h :$$
$$-W_{\eta\zeta}^{(1)}\sin\phi_1\cdot h : -W_{\eta\zeta}^{(1)}\cos\phi_1\cdot h$$

(by eqs. 20).

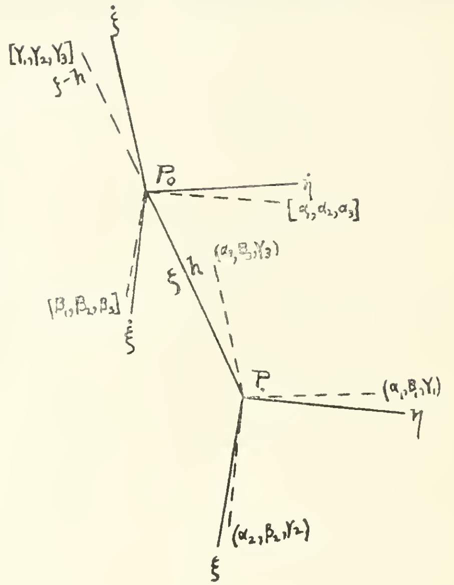

FIG. 5

$$\cos\alpha_2 : \cos\beta_2 : \cos\gamma_2 =$$

$$\begin{vmatrix} \cos\beta_3 & \cos\gamma_3 \\ \cos\beta_1 & \cos\gamma_1 \end{vmatrix} : \begin{vmatrix} \cos\gamma_3 & \cos\alpha_3 \\ \cos\gamma_1 & \cos\alpha_1 \end{vmatrix} : \begin{vmatrix} \cos\alpha_3 & \cos\beta_3 \\ \cos\alpha_1 & \cos\beta_1 \end{vmatrix}$$

$$= W_{\eta\zeta}^{(1)}\sin\phi_1\cdot h : W_{\zeta}^{(1)}\cos\phi_1 + (W_{\xi\zeta}^{(1)}\sin\phi_1 + 2W_{\zeta\zeta}^{(1)}\cos\phi_1)\,h :$$
$$-W_{\xi\zeta}^{(1)}\cos\phi_1\cdot h \;.$$

From these proportions the relations (26) easily follow.

We are now able to express the derivatives of W with respect to η, ξ, ζ in terms of the derivatives of W with respect to $\dot\eta$, $\dot\xi$, $\dot\zeta$. Thus

$$W_{\zeta} = W_{\dot\eta}\frac{\partial\dot\eta}{\partial\zeta} + W_{\dot\xi}\frac{\partial\dot\xi}{\partial\zeta} + W_{\dot\zeta}\frac{\partial\dot\zeta}{\partial\zeta} = W_{\dot\eta}\cos\gamma_1 + W_{\dot\xi}\cos\gamma_2 + W_{\dot\zeta}\cos\gamma_3$$

and

$$
\begin{aligned}
W_{\eta\zeta} = {} & (W_{\dot\eta\dot\eta}\cos\gamma_1 + W_{\dot\xi\dot\eta}\cos\gamma_2 + W_{\dot\zeta\dot\eta}\cos\gamma_3)\cos\alpha_1 \\
& + (W_{\dot\eta\dot\xi}\cos\gamma_1 + W_{\dot\xi\dot\xi}\cos\gamma_2 + W_{\dot\zeta\dot\xi}\cos\gamma_3)\cos\alpha_2 \\
& + (W_{\dot\eta\dot\zeta}\cos\gamma_1 + W_{\dot\xi\dot\zeta}\cos\gamma_2 + W_{\dot\zeta\dot\zeta}\cos\gamma_3)\cos\alpha_3
\end{aligned}
$$

Since $P_0-\dot\eta$, $\dot\xi$, $\dot\zeta$ are the cardinal axes of P_0,

$$W_{\dot\eta}^{0} = W_{\dot\xi}^{0} = 0$$

which, when solved for $\bar{t}$, gives the relation

$$\bar{t} = \left(\pm\sqrt{\frac{h}{a}}\right) - \tfrac{1}{2}b\left(\pm\sqrt{\frac{h}{a}}\right)^3 - \tfrac{1}{2}c\left(\pm\sqrt{\frac{h}{a}}\right)^4 + \quad ,$$

where the upper sign must be used because $\bar{t}$ is positive. Putting this value of $\bar{t}$ in equations (23) we get the equations

(33)

$$\left\{\begin{aligned}
\bar{\eta} &= \tfrac{2}{3}\sqrt{2}\,\omega\cos\phi_1 \frac{h^{3/2}}{\sqrt{-W_{\zeta}^{(1)}}} + \tfrac{5}{6}W_{\eta\zeta}^{(1)}\frac{h^2}{-W_{\zeta}^{(1)}} \\
&\quad - \frac{\sqrt{2}}{30}\,\omega\left\{18\sin\phi_1 \quad W_{\xi\zeta}^{(1)} - \cos\phi_1\cdot\left[7W_{\zeta\zeta}^{(1)}\right.\right. \\
&\quad \left.\left. + 2W_{\eta\eta}^{(1)} + 4\omega^2(5\cos^2\phi_1 - 2)\right]\right\}\frac{h^{5/2}}{[-W_{\zeta}^{(1)}]^{3/2}}, \\
\bar{\xi} &= \tfrac{1}{6}[5W_{\xi\zeta}^{(1)} + 4\omega^2\sin\phi_1\cos\phi_1]\,\frac{h^2}{-W_{\zeta}^{(1)}} \\
&\quad + \frac{\sqrt{2}}{15}\,\omega\left[9\sin\phi_1\cdot W_{\eta\zeta}^{(1)} + \cos\phi_1\,.\,W_{\xi\eta}^{(1)}\right]\frac{h^{5/2}}{[-W_{\zeta}^{(1)}]^{3/2}},
\end{aligned}\right.$$

and hence

(27)

$$\left\{\begin{aligned}
&W_{\zeta}^{\,0} = W_{\dot{\zeta}}^{\,0} + \text{terms in } h^2 \text{ and higher powers of } h\,. \\
&\text{also} \\
&W_{\zeta\eta}^{\,0} = W_{\dot{\zeta}\dot{\eta}}^{\,0} + \text{terms in } h \text{ and higher powers of } h\,. \\
&\text{Similarly} \\
&W_{\zeta\xi}^{\,0} = W_{\dot{\zeta}\dot{\xi}}^{\,0} + \text{terms in } h \text{ and higher powers of } h\,, \\
&W_{\zeta\zeta}^{\,0} = W_{\dot{\zeta}\dot{\zeta}}^{\,0} + \text{terms in } h \text{ and higher powers of } h\,, \\
&W_{\eta\eta}^{\,0} = W_{\dot{\eta}\dot{\eta}}^{\,0} + \text{terms in } h \text{ and higher powers of h}\,.
\end{aligned}\right.$$

By means of these relations and relations (20) we can express the values at P_1 of the derivatives of W with respect to η, ξ, ζ in terms of the values at P_0 of the derivatives of W with respect to $\dot{\eta}$, $\dot{\xi}$, $\dot{\zeta}$. Thus

(28)

$$\begin{aligned}
W_{\zeta}^{(1)} &= W_{\zeta}^{\,0} - W_{\zeta\zeta}^{\,0}\cdot h = W_{\dot{\zeta}}^{\,0} - W_{\dot{\zeta}\dot{\zeta}}^{\,0}\ h \\
&\qquad + \text{terms in } h^2 \text{ and higher powers of } h, \\
W_{\zeta\eta}^{(1)} &= W_{\zeta\eta}^{\,0} - W_{\zeta\eta\zeta}^{\,0}\cdot h = W_{\dot{\zeta}\dot{\eta}}^{\,0} \\
&\qquad + \text{terms in } h \text{ and higher powers of } h, \\
W_{\zeta\xi}^{(1)} &= W_{\zeta\xi}^{\,0} - W_{\zeta\xi\zeta}^{\,0}\cdot h = W_{\dot{\zeta}\dot{\xi}}^{\,0} \\
&\qquad + \text{terms in } h \text{ and higher powers of } h, \\
W_{\zeta\zeta}^{(1)} &= W_{\zeta\zeta}^{\,0} - W_{\zeta\zeta\zeta}^{\,0}\cdot h = W_{\dot{\zeta}\dot{\zeta}}^{\,0} \\
&\qquad + \text{terms in } h \text{ and higher powers of } h, \\
W_{\eta\eta}^{(1)} &= W_{\eta\eta}^{\,0} - W_{\eta\eta\zeta}^{\,0}\cdot h = W_{\dot{\eta}\dot{\eta}}^{\,0} \\
&\qquad + \text{terms in } h \text{ and higher powers of } h.
\end{aligned}$$

Hence equations (24) may be written as follows :

(29)

$$\begin{aligned}
\cos\phi_1 &= \cos\phi_0 - \sin\phi_0\cdot\frac{W_{\dot{\xi}\dot{\zeta}}^{\,0}}{W_{\dot{\zeta}}^{\,0}}\,h\,, \\
\sin\phi_1 &= \sin\phi_0 + \cos\phi_0\cdot\frac{W_{\dot{\xi}\dot{\zeta}}^{\,0}}{W_{\dot{\zeta}}^{\,0}}\,h\,.
\end{aligned}$$

Since

$$-W_{\zeta}^{(1)} = g_1\,, \qquad W_{\eta\zeta}^{(1)} = -\left(\frac{\partial g}{\partial\eta}\right)_1, \qquad W_{\xi\zeta}^{(1)} = -\left(\frac{\partial g}{\partial\xi}\right)_1,$$

$$W_{\zeta\zeta}^{(1)} = -\left(\frac{\partial g}{\partial\zeta}\right)_1, \qquad W_{\eta\eta}^{(1)} = \left(\frac{\partial W_{\eta}}{\partial\eta}\right)_1$$

equations (33) may be written as follows : (34)

$$\left.\begin{aligned}
\bar{\eta} &= \tfrac{2}{3}\sqrt{2}\,\omega\cos\phi_1\frac{h^{3/2}}{g_1^{1/2}} - \tfrac{5}{6}\frac{(\partial g/\partial\eta)_1}{g_1}h^2 \\
&\quad + \frac{\sqrt{2}}{30}\,\omega\left\{18\sin\phi_1\cdot\left(\frac{\partial g}{\partial\xi}\right)_1 - \cos\phi_1\cdot\left[7\left(\frac{\partial g}{\partial\zeta}\right)_1\right.\right. \\
&\quad \left.\left. - 2\left(\frac{\partial W_{\eta}}{\partial\eta}\right)_1 - 4\omega^2(5\cos^2\phi_1 - 2)\right]\right\}\frac{h^{5/2}}{g_1^{3/2}}, \\
\bar{\xi} &= \tfrac{1}{6}\left[4\omega^2\sin\phi_1\cos\phi_1 - 5(\partial g/\partial\xi)_1\right]\frac{h^2}{g_1} \\
&\quad + \frac{\sqrt{2}}{15}\,\omega\left[\cos\phi_1\cdot\left(\frac{\partial W_{\xi}}{\partial\eta}\right)_1 - 9\sin\phi_1\cdot\left(\frac{\partial g}{\partial\eta}\right)_1\right]\frac{h^{5/2}}{g_1^{3/2}}\,.
\end{aligned}\right\} *$$

If we now substitute in equation (22) the value given by relations (28) and (29), we obtain the equation

(30)

$$\begin{aligned}
h &= \frac{-\frac{1}{2}W_{\dot{\zeta}}^{\,0}\bar{t}^2 + \frac{1}{24}W_{\dot{\zeta}}^{\,0}[5W_{\dot{\zeta}\dot{\zeta}}^{\,0} + 4\omega^2\cos^2\phi_0]\bar{t}^4 + \frac{1}{6}\omega W_{\dot{\zeta}}^{\,0}W_{\dot{\eta}\dot{\zeta}}^{\,0}\cos\phi_0\cdot\bar{t}^5}{1 - \frac{1}{2}W_{\dot{\zeta}\dot{\zeta}}^{\,0}\bar{t}^2 + \cdots} \\
&= -\tfrac{1}{2}W_{\dot{\zeta}}^{\,0}\bar{t}^2 - \tfrac{1}{24}W_{\dot{\zeta}}^{\,0}[W_{\dot{\zeta}\dot{\zeta}}^{\,0} - 4\omega^2\cos^2\phi_0]\bar{t}^4 + \tfrac{1}{6}\omega W_{\dot{\zeta}}^{\,0}W_{\dot{\eta}\dot{\zeta}}^{\,0}\cos\phi_0\cdot\bar{t}^5
\end{aligned}$$

The same substitution makes the first of equations (23) take the form:

$$\begin{aligned}
\bar{\eta} = -\tfrac{1}{3}\omega\cos\phi_0\ W_{\dot{\zeta}}^{\,0}\bar{t}^3 &+ \tfrac{1}{3}\omega\sin\phi_0 W_{\dot{\xi}\dot{\zeta}}^{\,0}\,h\,\bar{t}^3 + \tfrac{1}{3}\omega\cos\phi_0 W_{\dot{\zeta}\dot{\zeta}}^{\,0}\,h\,\bar{t}^3 \\
&- \tfrac{5}{24}W_{\dot{\zeta}}^{\,0}W_{\dot{\eta}\dot{\zeta}}^{\,0}\bar{t}^4 \\
+ \tfrac{1}{60}\,\omega W_{\dot{\zeta}}^{\,0}[9\sin\phi_0\ W_{\dot{\xi}\dot{\zeta}} &+ \cos\phi_0(9W_{\dot{\zeta}\dot{\zeta}}^{\,0} - W_{\dot{\eta}\dot{\eta}}^{\,0} + 4\omega^2)]\bar{t}^5,
\end{aligned}$$

which, by (30), becomes

$$\left.\begin{aligned}
\bar{\eta} &= -\tfrac{1}{3}\omega\cos\phi_0\ W_{\dot{\zeta}}^{\,0}\bar{t}^3 - \tfrac{5}{24}W_{\dot{\zeta}}^{\,0}W_{\dot{\eta}\dot{\zeta}}^{\,0}\bar{t}^4 \\
&\quad - \tfrac{1}{60}\omega W_{\dot{\zeta}}^{\,0}[\sin\phi_0\ W_{\dot{\xi}\dot{\zeta}}^{\,0} + \cos\phi_0(W_{\dot{\zeta}\dot{\zeta}}^{\,0} + W_{\dot{\eta}\dot{\eta}}^{\,0} - 4\omega^2)]\bar{t}^5, \\
&\text{and the second of equations (23),} \\
\bar{\xi} &= -\tfrac{1}{24}W_{\dot{\zeta}}^{\,0}[5W_{\dot{\xi}\dot{\zeta}}^{\,0} + 4\omega^2\sin\phi_0\cos\phi_0]\bar{t}^4 \\
&\quad - \tfrac{1}{60}\omega W_{\dot{\zeta}}^{\,0}[9\sin\phi_0\cdot W_{\dot{\eta}\dot{\zeta}}^{\,0} + \cos\phi_0\cdot W_{\dot{\xi}\dot{\eta}}^{\,0}]\bar{t}^5\,.
\end{aligned}\right\}\ (31)$$

In equations (30) and (31) the coefficients of the powers of $\bar{t}$ are expressed in terms of the astronomical latitude of P_0 and the values at P_0 of the derivatives of W taken along the cardinal axes of P_0. These are the equations which were derived by the author in his second paper (see *T.A.M.S.*, Vol. XIII, p. 473, § 1, eqs. I), where η and $\bar{\xi}$ have the same meaning as in this paper, and $\bar{\zeta}$ stands for $-h$, ϕ for ϕ_0, t for $\bar{t}$ and η, ξ, ζ for $\dot{\eta}$, $\dot{\xi}$, $\dot{\zeta}$ respectively.

It is very gratifying to the author that the purely analytic proof given above leads to the same equations, even to the terms of highest order, as the quasi-geometric proof of the paper just referred to.

*By means of relations (28) and (29), the equations (33) assume the forms :

Equations (22) and (23), which express the height of fall h, the easterly deviation η and the southerly deviation $\bar{\xi}$ in terms of the time of fall t, and equations (33) and (34), which express $\bar{\eta}$ and $\bar{\xi}$ in terms of h, hold for an asymmetric distribution.

If, in particular, the distribution is one of revolution,

$$W_{\xi}^{(1)} = 0 \;, \qquad W_{\zeta}^{(1)} = 0 \;,$$

and therefore certain terms in these equations drop out. For instance, the second terms in the right hand members of each of the equations (34) become zero.

Remark. The coefficients of the powers of t in equations (22) and (23), and those of the powers of h in equations (33) and (34), are expressed in terms of the constants of P_1 (*i.e.* in terms of the astronomical latitude of P_1 and the values at P_1 of the derivatives of W taken along the cardinal axes of P_1). In equations (30), (31), (35), (36) of the two preceding footnotes the coefficients are expressed in terms of the constants of P_0. It is interesting to note that the coefficients of the first two terms in the developments (31), (35), (36) are built up from the constants of P_0 in exactly the same way as the coefficients of the first two terms in the developments (23), (33), (34), are built up from the constants of P_1.

(35)

$$\left\{\begin{aligned} \eta &= \tfrac{2}{3}\sqrt{2}\,\omega\cos\phi_0\,\frac{h^{3/2}}{\sqrt{-W_{\zeta}^{0}}} + \tfrac{5}{6}W_{\eta\zeta}^{0}\,\frac{h^2}{-W_{\zeta}^{0}} \\ &\quad + \frac{\sqrt{2}}{30}\,\omega\,\Big\{2\sin\phi_0\,W_{\xi\zeta}^{0} + 2\cos\phi_0\cdot W_{\zeta\zeta}^{0} - 3\cos\phi_0\,W_{\zeta}^{0} \\ &\quad + 4\omega^2(5\cos^3\phi_0 - 2\cos\phi_0)\Big\}\,\frac{h^{5/2}}{[-W_{\zeta}^{0}]^{3/2}} \\ \bar{\xi} &= \tfrac{1}{6}[5W_{\xi\zeta}^{0} + 4\omega^2\sin\phi_0\cos\phi_0]\,\frac{h^2}{-W_{\zeta}^{0}} \\ &\quad + \frac{\sqrt{2}}{15}\,\omega\left[9\sin\phi_0\cdot W_{\eta\zeta}^{0} + \cos\phi_0\,W_{\xi\eta}^{0}\right]\frac{h^{5/2}}{[-W_{\zeta}^{0}]^{3/2}} \end{aligned}\right.$$

These are equations II in the author's second paper (*T.A.M.S.*, Vol. XIII, p. 474), and may be written :

(36)

$$\left\{\begin{aligned} \eta &= \tfrac{2}{3}\sqrt{2}\,\omega\cos\phi_0\,\frac{h^{3/2}}{g_0^{1/2}} - \frac{5}{6}\,\frac{(\partial g/\partial\eta)_0}{g_0}\,h^2 \\ &\quad + \frac{\sqrt{2}}{30}\,\omega\,\Big\{3\cos\phi_0\cdot\left(\frac{\partial g}{\partial\zeta}\right)_0 - 2\sin\phi_0\cdot\left(\frac{\partial g}{\partial\xi}\right)_0 \\ &\quad + 2\left(\frac{\partial W_{\xi}}{\partial\eta}\right)_0\cos\phi_0 + 4\omega^2(5\cos^3\phi_0 - 2\cos\phi_0)\Big\}\,\frac{h^{5/2}}{g_0^{3/2}} \\ \xi &= \tfrac{1}{6}[4\omega^2\sin\phi_0\cos\phi_0 - 5(\partial g/\partial\xi)_0]\,\frac{h^2}{g_0} \\ &\quad + \frac{\sqrt{2}}{15}\,\omega\left[\cos\phi_0\left(\frac{\partial W_{\xi}}{\partial\eta}\right)_0 - 9\sin\phi_0\left(\frac{\partial g}{\partial\eta}\right)_0\right]\frac{h^{5/2}}{g_0^{3/2}}\;, \end{aligned}\right.$$

which are equations III in the paper just referred to. See statement below equations (31) in last footnote.

§4. The Algebraic Sign of the Meridional or Southerly Deviation and the Inadequacy of any Formula for this Deviation which Depends upon a Particular Potential Function.

From the second of equations (34) we see that the algebraic sign of the southerly deviation $\bar{\xi}$ is the same as that of the expression

$$2\,\omega^2\sin 2\,\phi_1 - 5\,(\partial g/\partial\xi)_1 \quad .$$

This expression is positive when $(-\,\partial g/\partial\xi)_1$ is positive, and negative when $(-\,\partial g/\partial\xi)_1$ is negative and

$$|-5\,(\partial g/\partial\xi)_1| > 2\omega^2\sin 2\,\phi_1 \quad .$$

In order to determine the direction of curvature of the lines of force of the weight field we will make use of the following well-known relations in Geodesy. *The curvature at P_1 of the line of force of the weight field* which passes through P_1 is given by the formula*

$$\frac{\sqrt{\left(\dfrac{\partial g}{\partial\xi}\right)_1^2 + \left(\dfrac{\partial g}{\partial\eta}\right)_1^2}}{g_1}\;,$$

and the direction from P_1 of the center of this curvature is given by the horizontal vector of components

$$\left(\frac{\partial g}{\partial\xi}\right)_1 \;, \qquad \left(\frac{\partial g}{\partial\eta}\right)_1 .$$

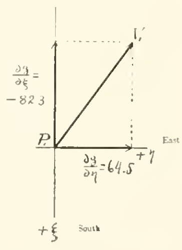

Fig. 6

Note. In the paper just referred to the following typographical errors occur. In formulas II and III, in the expressions for η and in the coefficient of $h^{5/2}$, $\cos^2\phi$ should be replaced by $\cos\phi$. Also in the first of these expressions for η and in the coefficient of h^2, $W_{\eta\xi}^{0}$ should be replaced by $W_{\eta\zeta}^{0}$. The reader can easily verify this by starting with formulas I which are correct.

*The differential equations of these lines of force are

$$\frac{du}{W_u} = \frac{dv}{W_v} = \frac{dz}{W_z} \quad \text{with respect to the axes } O - u,\, v,\, z \quad \text{and}$$

$$\frac{d\eta}{W_\eta} = \frac{d\xi}{W_\xi} = \frac{d\zeta}{W_\zeta} \quad \text{with respect to the axes } P_1 - \eta,\, \xi,\, \zeta \quad .$$

Thus for the values given in the following Table II for station 2,192, the direction of concavity is represented by the vector $P_1 V_1$ in Fig. 6. *For a distribution of revolution*

$$\left(\frac{\partial g}{\partial \eta}\right)_1 = 0 ,$$

and hence for this case the above expression for *the curvature assumes the form*

$$\left(-\frac{\partial g}{\partial \xi}\right)_1 \Big/ g_1 ,$$

the concavity being away from or toward the equator according as $(-\partial g/\partial \xi)_1$ *is positive or negative.**

Thus we see that the direction of curvature of the lines of force of the weight field, as well as the algebraic sign of the meridional deviation, depends upon the algebraic sign of the derivative $(-\partial g/\partial \xi)_1$.

The algebraic sign of $(-\partial g/\partial \xi)_1$ depends upon the form of the potential function W. No potential function is known which will fit all parts of the earth which lie near, and exterior to, the surface. A potential function which is sometimes used is the function†:

$$(38) \quad W = \frac{M\kappa}{r} + \frac{\kappa(C-A)}{2r^3}(1-3\sin^2\psi) + \frac{\omega^2}{2}r^2\cos^2\psi ,$$

where M is the mass of the earth, κ the gravitation constant, C and A the principal moments of inertia of the earth, C being that with respect to the axis of rotation and A that with respect to an equatorial axis. It is, of course, evident that this function corresponds to some distribution of revolution.

* This result may also be deduced from the relation

$$(37) \quad \phi_0 - \phi_1 = -\frac{W_{\xi\xi}^{(1)}}{W_\xi^{(1)}}\,h + \text{higher powers in } h$$

$$= -\frac{(\partial g/\partial \xi)_1}{g_1}\,h + \text{higher powers in } h ,$$

which is obtained from equations (24_1) and (24_2). In the first place this relation (37), which holds for a general asymmetric distribution, shows that *the astronomical latitude of* P_0 (*which is above* P_1 *and in the vertical of* P_1) *is greater than or less than that of* P_1 *according as* $(-\partial g/\partial \xi)$ is positive or negative.

Geometrical considerations will easily convince one that for a distribution of revolution *the limit of the ratio* $(\phi_0-\phi_1)/h$, which by the above relation (37) is equal to $(-\partial g/\partial \xi)_1/g_1$, is the curvature of the line of force of the weight field which passes through P_1.

†See Helmert, *die Mathematischen und Physikalischen Theorieen der Höheren Geodäsie*, II Teil, p. 75, or Poincaré, *Figures d'Équilibre d'une Masse Fluide* (1902) Chap. V.

According to Poincaré,*

$$C - A = \tfrac{2}{3}\left(e_1 - \frac{\sigma_1}{2}\right) M r_1^2 , \tag{39}$$

where r_1 and e_1 are the mean radius and ellipticity, respectively, of the standard spheroid :

$$r = r_1\left[1 + \frac{e_1}{3}(1 - 3\sin^2\psi)\right], \tag{40}$$

and

$$\sigma_1 = \frac{\omega^2 r_1}{\kappa M/r_1^2} .$$

The values of the constants are

$$e_1 = \frac{1}{293.5} = .003407 , \quad \sigma_1 = \frac{1}{288.38} = .003468 ,$$

and hence

$$\epsilon = e_1 - \frac{\sigma_1}{2} = .001673 .$$

Therefore†

$$C - A = \tfrac{2}{3}\epsilon M r_1^2 = .00111\,M r_1^2 \tag{39_1}$$

and the function (38) may be written in the form‡

$$W = f(\tau, z) = \frac{M\kappa}{r}\left[1 + \frac{\epsilon}{3}r_1^2\frac{\tau^2 - 2z^2}{r^4}\right] + \frac{\omega^2}{2}\tau^2 . \tag{41}$$

The derivatives of first order of this function are

$$-\frac{\partial f}{\partial \tau} = \frac{M\kappa\tau}{r^3}\left[1 + \epsilon r_1^2\frac{\tau^2 - 4z^2}{r^4}\right] + \omega^2\tau , \tag{42}$$

$$-\frac{\partial f}{\partial z} = \frac{M\kappa z}{r^3}\left[1 + \epsilon r_1^2\frac{3\tau^2 - 2z^2}{r^4}\right] .$$

Hence the acceleration g at any point (τ, z) is

$$g = \sqrt{\left(\frac{\partial f}{\partial \tau}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2} \tag{43}$$

$$= \frac{M\kappa}{r^2}\left[1 + \left(\frac{r_1}{r}\right)^2\frac{\tau^2 - 2z^2}{r^2}\,\epsilon - \frac{\tau^2}{r^2}\,\sigma + \cdots\right],$$

*Loc. cit. p. 106. The quantity r_1 used from equations (39) to (45) is *not* the length of the radius vector OP_1 (Fig. 2.)

†Dr. Woodward, in the last of his relations (31) (*A.J.*, Nos. 651-652, p. 23) gives $C - A = .001065\,Ma^2$, where a is the major semi-axis of the ellipsoid (40). Since by (40),

$$a = r_1\left(1 + \frac{e_1}{3}\right) ,$$

Dr. Woodward's value for $C - A$ does not differ much from that here given.

‡This is the potential function given by Dr. Woodward, who writes it in the form (38) in his equation (26) (where $B = A$ by his equations (31) and $V = W - \frac{\omega^2}{2}r^2\cos^2\psi$), and in a different form in his equation (46). It is also the potential function which I give in my first paper (*T.A.M.S.*, Vol. XII, pp. 335-353) under assumption 4, where M, ρ, ρ_1, r correspond to $M\kappa, r, r_1, \tau$ used in (41).

where

$$\sigma = \frac{\omega^2 r^3}{M_K} = \frac{\omega^2 r_1^3}{M_K}\left(\frac{r}{r_1}\right)^3 = \sigma_1[1 + \partial_1(1 - 3\sin^2\psi)] \quad ,$$

by (40), and

$$\frac{1}{r^2} = \frac{1}{a^2}\left(\frac{a}{r_1}\right)^2\left(\frac{r_1}{r}\right)^2$$

$$= \frac{1}{a^2}(1 + \tfrac{2}{3}\partial_1)[1 - \tfrac{2}{3}\partial_1(1 - 3\sin^2\psi)]$$

$$= \frac{1}{a^2}(1 + 2\partial_1\sin^2\psi) \,,$$

a being the major semi-axis of the spheroid (40). Therefore the value of g on the standard spheroid is

$$g = \frac{M_K}{a^2}(1 + 2\partial_1\sin^2\psi)[1 + \epsilon(1 - 3\sin^2\psi) - \sigma_1(1 - \sin^2\psi)] \tag{44}$$

$$= \frac{M_K}{a^2}(1 + \epsilon - \sigma_1)[1 + (2\sigma_1 - \epsilon)\sin^2\psi] \;.$$

By (42) and (43), $\frac{M_K}{a^2}(1 + \epsilon - \sigma_1)$ is the value of g at the equator, *i.e.* for $z = 0$ and $\tau = r = a$. If we denote this value of g by g_a, formula (44) takes the form*

$$g = g_a[1 + (2\sigma_1 - \epsilon)\sin^2\psi] \;. \tag{45}$$

and by the values given above

$$2\sigma_1 - \epsilon = .00526 \;.$$

From the last formula we get by differentiation.

$$-\frac{\partial g}{\partial \xi} = \frac{2\sigma_1 - \epsilon}{r} \cdot g_a \sin 2\psi \quad , \tag{46}$$

since

$$-d\xi = r\,d\psi \quad + \text{terms of higher order.}$$

If we assume

$$g_a = 978.06 \text{ cm.}$$

and

$$r_1 = \frac{2a + b}{3} = 637105100 \text{ cm.}$$

we find that

$$\frac{2\sigma_1 - \epsilon}{r} g_a = +8.1 \times 10^{-9} \tag{47}$$

From formulas (45) and (46) the values of g and $-\frac{\partial g}{\partial \xi}$ can be found for different latitudes. These values do not differ much from those which are given in the following table, which is computed by a formula due to Helmert. (See formula (48) this paper)

See Helmert, loc. cit. formulae (5) and (5)

Table I.*

ϕ	g	$-10^9\,\partial g/\partial\xi$
40°	980.1457	8.0399
45°	980.5966	8.1568
50°	981.0475	8.0259
55°	981.4847	7.6517
60°	981.8949	7.0463

By formula (46), in which the coefficient of $\sin 2\psi$ has the value (47), we see that for the potential function (38), the derivative $(-\partial g/\partial\xi)_1$ is positive, and therefore by the first two paragraphs of this section (§4) we conclude that *for the potential function* (38), (1) *the meridional deviation is toward the equator.* (2) *the lines of force of the weight field are convex to the equator.* (3) *the astronomical latitude of a point* P_0, *which is above* P_1 *and in the vertical of* P_1, *is numerically greater than that of* P_1.

While the conclusions just stated hold for the potential function (38), they may be entirely erroneous for another potential function. In fact, as has already been stated, no potential function is known which fits every region of the earth near, and exterior to, its surface. However, it is possible to determine experimentally the values of g and $(\partial g/\partial\xi)_1$ at any given place. The first quantity is determined by means of pendulum observations and the second by means of a torsion balance devised by the Hungarian physicist Baron Roland Eötvös.

Observation shows that, in a fairly level country, experimentally determined values of g do not differ much from those given by formula (45). Any local influence, such as a mountain or a mineral deposit, which causes the experimentally determined value of g to differ much from that given by formula (45), and the astronomical latitude and longitude to differ much from the geographic (*i.e.* those determined by the normal to the spheroid (40)), is regarded as a local irregularity in the earth's weight field. While the deviations from the normal values (*i.e.* those given by Table I) which are produced in the values of g by local irregularities in the weight field are relatively small, those produced in $(\partial g/\partial\xi)_1$ are relatively very great, as the following table will serve to indicate.

Table II.

Station Number	$-10^9\left(\frac{\partial g}{\partial \xi}\right)_1$	$+10^9\left(\frac{\partial g}{\partial \eta}\right)_1$	$-10^9\left(\frac{\partial^2 W}{\partial\xi\partial\eta}\right)_1$
1,018	+32.9	+88.1	+ 3.8
1,032	+25.8	+62.8	+ 2.5
1,035	+15.5	+69.8	− 4.6
2,159	−37.0	+34.5	−13.5
2,188	+77.8	+59.8	+30.3
2,192	+82.3	+64.5	+26.7

*This table is given in my second paper (*T.A.M.S.*, Vol. XIII, p. 170.)

The values given in this table were obtained in a survey which was made with the Eötvös balance in a region just east of Arad, in Hungary.* The stations, of which the numbers are given in the table, are just west of some mountains. The latitude of this region is about 46° 10′.

The value of g at Arad ($\varphi = 46° \; 10' \; 17''$) determined by pendulum observations, is $= 980.724$, the correction for height is $+.034$, thus making the sea-level value, $g = 980.758$. The theoretical value given by the formula

$$g_t = 978.030\,(1 + 0.005302 \sin^2 \phi - 0.000007 \sin^2 2\phi), \tag{48}$$

from which Table I was computed, is 980.721, thus making $g - g_t = +.037$. In similar determinations, made at ten other places in Hungary,† the difference $g - g_t$ was found to lie between $+.026$ and $+.049$. This result bears out the statement, made above Table II, that the deviations in g are relatively small.

On the other hand, the figures in Table II show that the deviations in $(-\partial g/\partial \xi)_1$, $(\partial g/\partial \eta)_1$, $(-\partial^2 W/\partial \xi \partial \eta)_1$ from the normal values $+8.1 \times 10^{-9}$, 0, 0†, which correspond to the potential function (38), are relatively very great. Thus at station 2,192, $(-\partial g/\partial \xi)_1$ is more than ten times as great as the normal value, while at station 2,159, $(-\partial g/\partial \xi)_1$ is negative and numerically four and one-half times as great as the normal value. Hence it follows from the general formula (κ) (Part I), the special formula (π_1) (Part I), which corresponds to the potential function (38), and the formula of Gauss ($S.D. = \frac{1}{3}\omega^2 \sin 2\phi \cdot h^2/g$), (*1*) that at station 2,192 the meridional (southerly) deviation of a freely falling body would be $8\frac{1}{2}$ times as great as that corresponding to the potential function (38) and 42 times as great as that given by the formula of Gauss; (*2*) that at station 2,159 the meridional deviation of a freely falling body would be *poleward, i.e., northerly*, and $-3\frac{1}{2}$ times that corresponding to the potential function (38) and $-17\frac{1}{2}$ times that given by the formula of Gauss.

These results are sufficient to prove the inadequacy of any formula for the meridional deviation of freely falling bodies which is based upon a particular potential function, and to show that *in the future, experiments for the determination of the deviations of freely falling bodies should be preceded by premilinary experiments for the determination of the local value of* $(-\partial g/\partial \xi)$.

When the experimentally determined local values of the derivatives $(\partial g/\partial \xi)_1$, $(\partial g/\partial \eta)_1$, $(\partial^2 W/\partial \xi \partial \eta)_1$ are substituted in formulas* (34) (which do *not* depend upon any particular potential function) the effect, on the deviations of a freely falling particle, of local irregularities in the Earth's field of force is taken into account. As already stated, these formulas do not take into account the effects of air resistance, air currents, the actions of the Moon and the Sun and the curvature of the string of the plumb-line.

See Eötvös, 1en Band der Abhandlungen der XV Allgemeinen Konferenz der Erdmessung in Budapest, 1906, pp. 337-395.

†See Appendix by Karl Oltay in the Eötvös paper, *XVI Allgemeine Konferenz der Internationale Erdmessung.*

†Since the potential function (38) corresponds to some distribution of revolution

$$\left(\frac{\partial^2 W}{\partial \zeta \, \partial \eta}\right)_1 = 0 \, , \qquad \left(\frac{\partial^2 W}{\partial \xi \, \partial \eta}\right)_1 = 0 \, .$$

*It should be noted that the values of the derivatives $(\partial g/\partial \zeta)_1 = (-\partial^2 W/\partial \zeta^2)_1$ and $(\partial^2 W/\partial \eta^2)_1$, which occur in the third term of the expression for the easterly deviation η, can not be determined by the Eötvös method.

TRANSIT OF *MERCURY*, 1914 NOV. 7.

By F. P. LEAVENWORTH.

The times of the third and fourth contacts were observed with the $10\frac{1}{2}$-in. Equatorial power 150 as follows: —

Third contact	2^h	7^m	28^s	Gr. M. T.
Fourth contact	2	9	25	Gr. M. T.

The observation was made through a light bank of clouds, and through the naked branches of a tree. But the definition was fair for the third contact. The fourth contact is more doubtful owing to the wavy appearance of the Sun's limb at that time.

University of Minnesota, Minneapolis.

EPHEMERIS OF ASTEROID "MT" (ALBERT),

By F. E. SEAGRAVE.

Epoch = April 30.50 = 1915, G. M. T.

$M = 317° 51' 22''.84$

$\pi = 337° 29' 19''.29$

$\Omega = 185° 32' 36''.92$

$i = 10° 49' 48''$

$\text{Log } e = 9.732842$

$\text{Log } q = 0.074722$

$\mu = 853''.66491$

CONSTANTS.

$x = r(9.99993) \sin(275° 26' 43''.60 + u)$

$y = r(9.98928) \sin(185° 12' 42''.00 + u)$

$z = r(9.34256) \sin(190° 3' 41''.10 + u)$

Gr. Midnight	α	δ	log r	log Δ
1915	h m s	° ′ ″		
April 26	16 11 33	−7 21 48	0.33616	0.09192
30	16 9 18	−6 36 57	0.33084	0.07573
May 4	16 6 33	−5 50 27	0.32542	0.06018
8	16 3 20	−5 2 16	0.31990	0.04552
12	15 59 38	−4 14 13	0.31430	0.03185
16	15 55 34	−3 26 46	0.30856	0.01932
20	15 51 10	−2 39 45	0.30274	0.00808
☍				
24	15 46 36	−1 54 35	0.29684	9.99826
28	15 41 50	−1 11 52	0.29082	9.98986
June 1	15 37 4	−0 32 30	0.28470	9.98291
5	15 32 23	+0 2 54	0.27846	9.97740
9	15 27 53	+0 33 38	0.27214	9.97331
13	15 23 41	+0 59 8	0.26572	9.97057

Opposition = May 21ᵈ 1915. Distance from Earth unity.
Nearest to Earth = June 19, 1915.
Perihelion = October 25, 1915.

OBSERVATIONS OF COMET 1913 *f* (*DELAVAN*),

MADE WITH THE FILAR MICROMETER OF THE 12½-IN. EQUATORIAL OF DETROIT OBSERVATORY

By BERNHARD H. DAWSON.

1914	Ann Arbor M.T	✻	Comp.	Δα	Δδ	α App.	log p Δ	δ App.	log p′Δ
	h m s			m s	′ ″	h m s		° ′ ″	
Oct. 21	6 52 43	1	10 , 8	+ 1 21.11	− 1 25.1	13 51 55.73	9.6874	+ 30 18 36.2	0.7727
26	6 36 12	2	10 , 10	− 0 38.09	+ 6 15.7	14 12 8.66	9.6788	+ 26 26 44.9	0.7637
Nov. 1	6 22 32	3	8 , 8	− 0 17.24	+ 4 37.1	14 33 30.99	9.6669	+ 21 54 56.5	0.7626

Mean Places of Comparison-Stars.

✻	α 1914.0	Red. to App. Pl.	δ 1914.0	Red. to App. Pl.	Authority
	h m s	s	° ′ ″	″	
1	13 50 33.28	+ 1.34	+ 30 20 15.9	− 14.6	A.G. Leiden 5040
2	14 12 45.38	+ 1.37	+ 26 20 44.0	− 14.8	A.G. Cambridge Eng., 6771.
3	14 33 46.81	+ 1.42	+ 21 50 34.1	− 14.7	A.G. Berlin B 5119.

The first observation was made by the method of micrometer transits, the others by direct micrometer measurement of Δα and Δδ. All the observations have been corrected for differential refraction.

Ann Arbor, 1914 *Nov.* 14.

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